Çankaya University – ECE Department – ECE 646 (FE)

Student Name : Date : 31.05.2012

Student Number : Open book exam

Questions

1. (70 Points) By taking the following receiver field expression of a Gaussian beam, derive and show that the scintillation index of the Gaussian beam becomes as given by $m^2\ \mathbf{r}, L$ in cylindrical coordinates. You can perform your derivation in cylindrical or Cartesian coordinates. Insert the given $m^2\ \mathbf{r}, L$ formulation into Scin_SinoHyp_L.m (to be used together with dblquade04.m) or into an m code you write and then produce the scintillation curve of Gaussian beam against propagation distance at the following source and propagation settings . Compare the values of $m^2\ \mathbf{r}, L$ at $L = 2$ and 3 km from your derivation with Fig. 3 of the article, "Scintillations characteristics of cosh-Gaussian beams", *Applied Optics*, **46**(7), 1099-1106 (2007), also available on the course web page as pdf text and Matlab fig file. Adopt the other settings as follows.

   a) $\alpha_{sx} = \alpha_{sy}$=1 cm (in Cartesian) or $\alpha_s = \sqrt{2}$ cm (in cylindrical), $F_s \rightarrow \infty$

   b) $C_n^2 = 10^{-15}$, $\lambda = 1.55\ \mu\text{m}$, von-Karman spectrum, $\Phi_n\ \kappa\ = \dfrac{\exp\left[-\ell_0^2\kappa^2/35.05\right]}{\kappa^2 + 4\pi^2/L_0^2\ ^{11/6}}$ with

   $\ell_0 = 1$ mm, $L_0 = 50$ m.

   c) At receiver plane coordinates of $r_x = r_y = 0.5$ cm (in Cartesian) or $r = 0.5\sqrt{2}$ cm

2. Send your figure file to h.eyyuboglu@cankaya.edu.tr address.

Solution : We provide the solution for both cylindrical and Cartesian coordinates. Gaussian beam in both coordinates and on source and receiver planes are shown below

$$u_s\left(s, \phi_s\right) = A_c \exp\left(-k\alpha s^2\right)$$ Gaussian beam (GB) source field in cylindrical coordinates

$$u_s\left(s_x, s_y\right) = A_c \exp\left[-0.5k\left(\alpha_x s_x^2 + \alpha_y s_y^2\right)\right]$$ Gaussian beam source field in Cartesian coordinates

$$u_r\left(r, \phi_r, z\right) = A_c \frac{\exp\left(jkz\right)}{1 + 2j\alpha z} \exp\left(-\frac{k\alpha r^2}{1 + 2j\alpha z}\right)$$ GB receiver field in cylindrical coordinates

$$u_r\left(r_x, r_y, z\right) = \frac{A_c \exp\left(jkz\right)}{\left(1 + j\alpha_x z\right)^{0.5}\left(1 + j\alpha_y z\right)^{0.5}} \exp\left[-\frac{k\alpha_x r_x^2}{2\left(1 + j\alpha_x z\right)} - \frac{k\alpha_y r_y^2}{2\left(1 + j\alpha_y z\right)}\right]$$ [GB receiver field in Cartesian coordinates]

Note that in the following derivations, we shall omit $\exp\left(jkz\right)$. To arrive at $m^2\ \mathbf{r}, L$ as defined in Notes of ECE 646_HTE_Bahar 2012, we initially define the following integrals

$$H(r,\phi_r,\kappa,\phi_\kappa,\eta)=\frac{k^2}{2\pi(L-\eta)u_r(r,\phi_r,z=L)}\exp\left[\frac{jkr^2}{2(L-\eta)}\right]\int_0^\infty\int_0^{2\pi}dr_1 d\phi_1 r_1 u_r(r_1,\phi_1,z=\eta)$$

$$\times\exp\left[j\kappa r_1\cos(\phi_1-\phi_\kappa)\right]\exp\left\{\frac{jk}{2(L-\eta)}\left[r_1^2-2r_1 r\cos(\phi_1-\phi_r)\right]\right\}\quad \text{in cylindrical coordinates}$$

$$H(r_x,r_y,\kappa,\phi_\kappa,\eta)=\frac{k^2\exp\left[jk(L-\eta)\right]}{2\pi(L-\eta)u_r(r_x,r_y,z=L)}\exp\left[\frac{jk(r_x^2+r_y^2)}{2(L-\eta)}\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dr_{1x}dr_{1y}u_r(r_{1x},r_{1y},z=\eta)$$

$$\times\exp\left(j\kappa\left[r_{1x}\cos(\phi_\kappa)+r_{1y}\sin(\phi_\kappa)\right]\right)\exp\left[\frac{jk}{2(L-\eta)}\left(r_{1x}^2+r_{1y}^2-2r_x r_{1x}-2r_y r_{1y}\right)\right]\quad \text{in Cartesian coordinates}$$

After inserting for $u_r(r,\phi_r,z)$ in $H(r,\phi_r,\kappa,\phi_\kappa,\eta)$, to solve the integration over ϕ_1, we use the following

$$\int_0^{2\pi}dx\exp(jp\cos x+jq\sin x)=2\pi J_0\left(\sqrt{p^2+q^2}\right)\qquad \text{(UF1)}$$

By arranging the integration over ϕ_1 in $H(r,\phi_r,\kappa,\phi_\kappa,\eta)$ as in (UF1), we get

$$I_{\phi_1}=\int_0^{2\pi}d\phi_1\exp\left[j\left(\kappa r_1\cos\phi_\kappa-\frac{k}{L-\eta}rr_1\cos\phi_r\right)\cos\phi_1+j\left(\kappa r_1\sin\phi_\kappa-\frac{k}{L-\eta}rr_1\sin\phi_r\right)\sin\phi_1\right]$$

This way p, q and p^2+q^2 will be

$$p=\kappa\cos\phi_\kappa-\frac{k}{L-\eta}r\cos\phi_r\quad,\quad q=\kappa\sin\phi_\kappa-\frac{k}{L-\eta}r\sin\phi_r$$

$$p^2+q^2=\kappa^2+\frac{k^2r^2}{(L-\eta)^2}-\frac{2k\kappa r}{L-\eta}\cos(\phi_\kappa-\phi_r)$$

So I_{ϕ_1} will be

$$I_{\phi_1}=2\pi J_0\left(r_1\sqrt{p^2+q^2}\right)=2\pi J_0\left\{r_1\left[\kappa^2+\frac{k^2r^2}{(L-\eta)^2}-\frac{2k\kappa r}{L-\eta}\cos(\phi_\kappa-\phi_r)\right]^{0.5}\right\}$$

Now for the remaining integration over r_1 in $H(r,\phi_r,\kappa,\phi_\kappa,\eta)$, we use the following formulation

$$\int_0^\infty dx x^{\nu+1}\exp(-ax^2)J_\nu(\beta x)=\frac{\beta^\nu}{(2a)^{\nu+1}}\exp\left(-\frac{\beta^2}{4a}\right)$$

So I_{r_1} is

$$I_{r_1} = \int_0^{\infty} dr_1 r_1 \exp\left[-\left(\frac{k\alpha}{1+2j\alpha\eta} - \frac{jk}{2(L-\eta)}\right) r_1^2\right] J_0\left(\left\{\left[\kappa^2 + \frac{k^2 r^2}{(L-\eta)^2} - \frac{2k\kappa r}{L-\eta}\cos(\phi_\kappa - \phi_r)\right]^{0.5}\right\}\right)$$

$$= \frac{\beta^{\nu}}{(2a)^{\nu+1}} \exp\left(-\frac{\beta^2}{4a}\right)$$

On the second line of above equation

$$\nu = 0,\ a = \frac{k\alpha}{1+2j\alpha\eta} - \frac{jk}{2(L-\eta)} = -jk\frac{1+2j\alpha\eta}{2(1+2j\alpha\eta)(L-\eta)},$$

$$\beta = \left[\kappa^2 + \frac{k^2 r^2}{(L-\eta)^2} - \frac{2k\kappa r}{L-\eta}\cos(\phi_\kappa - \phi_r)\right]^{0.5}$$

Now collect amplitude terms denoted by AF

$$AF = \frac{k^2}{2\pi(L-\eta)} 2\pi \frac{A_c}{1+2j\alpha\eta} j\frac{(1+2j\alpha\eta)(L-\eta)}{k(1+2j\alpha L)} = jk\frac{A_c}{1+2j\alpha L}$$

Note that eventually $\frac{A_c}{1+2j\alpha L}$ of AF this will cancel with $u_r(r,\phi_r,z=L)$ in $H(r,\phi_r,\kappa,\phi_\kappa,\eta)$.
Now collect all exponential terms

$$\exp\left[\frac{jkr^2}{2(L-\eta)}\right]\exp\left(-\frac{\beta^2}{4a}\right) = \exp\left[\frac{jkr^2}{2(L-\eta)} - j\frac{\kappa^2(1+2j\alpha\eta)(L-\eta)}{2k(1+2j\alpha L)} - j\frac{kr^2(1+2j\alpha\eta)}{2(1+2j\alpha L)(L-\eta)}\right.$$
$$\left. + j\frac{\kappa r(1+2j\alpha\eta)}{1+2j\alpha L}\cos(\phi_\kappa - \phi_r)\right] = \exp\left[-\frac{k\alpha r^2}{1+2j\alpha L} - j\frac{\kappa^2(1+2j\alpha\eta)(L-\eta)}{2k(1+2j\alpha L)} + j\frac{\kappa r(1+2j\alpha\eta)}{1+2j\alpha L}\cos(\phi_\kappa - \phi_r)\right]$$

Note that the first term of the exponential on the second line will further cancel with exponential of $u_r(r,\phi_r,z=L)$ in $H(r,\phi_r,\kappa,\phi_\kappa,\eta)$.
Finally collecting all terms $H(r,\phi_r,\kappa,\phi_\kappa,\eta)$ will be

$$H(r,\phi_r,\kappa,\phi_\kappa,\eta) = jk\exp\left[-j\frac{\kappa^2(1+2j\alpha\eta)(L-\eta)}{2k(1+2j\alpha L)} + j\frac{\kappa r(1+2j\alpha\eta)}{1+2j\alpha L}\cos(\phi_\kappa - \phi_r)\right]$$

Note that $H(r,\phi_r,\kappa,\phi_\kappa,\eta)$ agrees perfectly with (5) of the article "Scintillation advantages of lowest order Bessel-Gaussian beams", *Applied Physics B - Laser and Optics*, 92(2), 229-235 (2008).
Now to arrive at $m^2(\mathbf{r},L)$, we have to evaluate $H(r,\phi_r,\kappa,\phi_\kappa,\eta)H^*(r,\phi_r,\kappa,\phi_\kappa,\eta)$ and $H(r,\phi_r,\kappa,\phi_\kappa,\eta)H(r,\phi_r,-\kappa,\phi_\kappa,\eta)$. These are found to be

$$H(r,\phi_r,\kappa,\phi_\kappa,\eta)\,H^*(r,\phi_r,\kappa,\phi_\kappa,\eta) = k^2 \exp\left\{-\frac{\kappa^2\left[(\alpha+\alpha^*)(L-\eta)\right]^2}{k\left[1+2j(\alpha-\alpha^*)L+4|\alpha|^2 L^2\right]}\right\}$$
$$\times\exp\left[\frac{2\kappa r(\alpha+\alpha^*)(L-\eta)}{1+2j(\alpha-\alpha^*)L+4|\alpha|^2 L^2}\cos(\phi_\kappa-\phi_r)\right]$$

$$H(r,\phi_r,\kappa,\phi_\kappa,\eta)\,H(r,\phi_r,-\kappa,\phi_\kappa,\eta) = -k^2\exp\left[-\frac{j\kappa^2(1+2j\alpha\eta)(L-\eta)}{k(1+2j\alpha L)}\right]$$

Now $m^2(\mathbf{r},L)$ will be

$$m^2(\mathbf{r},L) = m^2(r,\phi_r,L) = 4\pi\int_0^L d\eta\int_0^{2\pi} d\phi_\kappa\int_0^\infty d\kappa\kappa\,\Phi_n(\kappa)$$
$$\times\left\{|H(r,\phi_r,\kappa,\phi_\kappa,\eta)|^2+\mathrm{Re}\left[H(r,\phi_r,\kappa,\phi_\kappa,\eta)\,H(r,\phi_r,-\kappa,\phi_\kappa,\eta)\right]\right\}$$

In the above, the integration over ϕ_κ in $H(r,\phi_r,\kappa,\phi_\kappa,\eta)\,H^*(r,\phi_r,\kappa,\phi_\kappa,\eta)$ can be managed using

$$\int_0^{2\pi} d\phi_\kappa \exp\left[C(\cos\phi_r\cos\phi_\kappa+\sin\phi_r\sin\phi_\kappa)\right] = 2\pi I_0(C) \qquad \text{(UF3)}$$

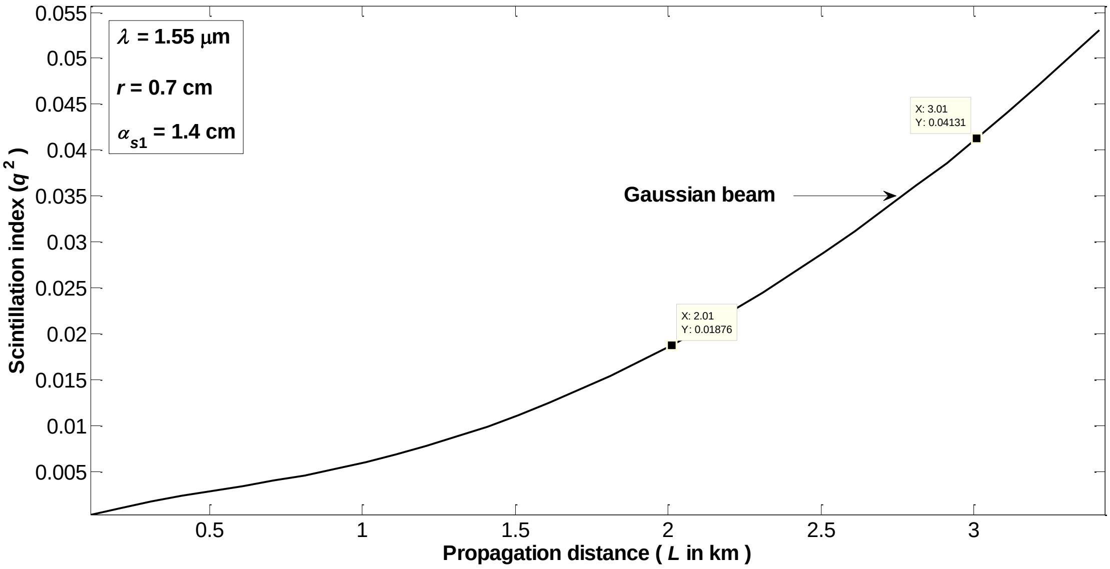
where

$$C = \frac{2\kappa r(\alpha+\alpha^*)(L-\eta)}{1+2j(\alpha-\alpha^*)L+4|\alpha|^2 L^2}$$

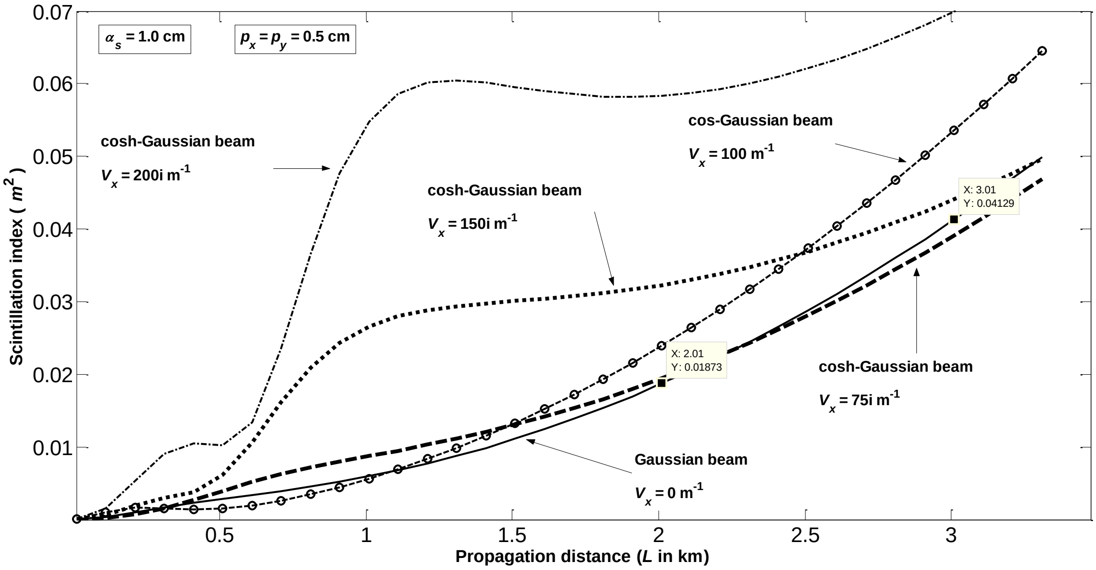
Eventually by setting the spectrum function to von-Karman $m^2(r,\phi_r,L)$ as a double integral will become

$$m^2(r,\phi_r,L) = 2.6056k^2C_n^2\int_0^L d\eta\int_0^\infty d\kappa\kappa\frac{\exp\left[-\ell_0^2\kappa^2/35.05\right]}{\left(\kappa^2+4\pi^2/L_0^2\right)^{11/6}}\left(\exp\left\{-\frac{\kappa^2\left[(\alpha+\alpha^*)(L-\eta)\right]^2}{k\left[1+2j(\alpha-\alpha^*)L+4|\alpha|^2L^2\right]}\right\}\right.$$
$$\left.\times I_0\left[\frac{2\kappa r(\alpha+\alpha^*)(L-\eta)}{1+2j(\alpha-\alpha^*)L+4|\alpha|^2L^2}\right]-\mathrm{Re}\left\{\exp\left[-\frac{j\kappa^2(1+2j\alpha\eta)(L-\eta)}{k(1+2j\alpha L)}\right]\right\}\right)$$

By running Scin_SinoHyp_L.m at the given parameter settings for Gaussian beam (or by creating a new m file using the above derived scintillation index formulation), we get the following figure



On the other hand, Fig. 3 from the article, "Scintillations characteristics of cosh-Gaussian beams", *Applied Optics*, **46**(7), 1099-1106 (2007) is also given below with markings at $L = 2.01$ km, $L = 3.01$ km (note that we have used these fractional values due to the specific increment settings)



As seen, the scintillation index (i.e. Y) values in the two figures differ only in the fourth digits which can be neglected.

2. (30 Points) Answer the following questions as **True** or **False.** For the **False** ones give the correct answer or the reason. For the **True** ones, justify your answer

a) Scintillation can be found only by applying Rytov method : False, since scintillation can also be found using Hygens-Fresnel (HF) integrals.

b) Scintillation results from atmospheric turbulence : True, for instance, there is no scintillation in vacuum, since there is no turbulence there.

c) A fully coherent source becomes partially coherent as it propagates in turbulent atmosphere : True, as proven in complex degree of coherence experiment. This is because turbulence changes and gradually destroys the spatial and temporal relationship in the source coherence.

d) Mutual coherence function is generally complex : True, since mutual coherence function measures the correlation between two distinct locations of the receiver plane.

e) If displacement parameter is set to zero, then from **Sinusoidal / Hyperbolic Gaussian beam** formulation, we always obtain Gaussian beam : Not exactly, since in annular Gaussian beam, the displacement parameter is also zero.

f) Scintillation formulation via Huygens Fresnel integral contains the average of squared intensity : True, since in this case scintillation index is obtained by dividing the average of squared intensity by the square of the average intensity.