Notes of ECE 646 : Theoretical Formulations and Simulations HTE – 21.02.2012

1. Introduction

In atmosphere near the ground due to changes in wind velocity and temperature, refractive index changes by small amounts causing what is called turbulence. These changes will give rise to fluctuations in amplitude and phase of the optical wave propagating in such a medium. Since there is randomness, we characterize propagation through atmosphere by statistical quantities to be added to the usual Huygens – Fresnel integral.

We remind that in free space, where there are no refractive index changes with respect to spatial or temporal coordinates, the Huygens Fresnel integral reads

$$u_r(r,\phi_r,z) = \frac{-jk\exp(jkz)}{2\pi z}\int_0^{\infty}\int_0^{2\pi} ds\, d\phi_s\, s\, u_s(s,\phi_s)\exp\left\{\frac{jk}{2z}\left[-2rs\cos(\phi_r-\phi_s)+s^2+r^2\right]\right\} \quad \text{in cylindrical coordinates} \quad (1.1)$$

$$u_r(r_x,r_y,z) = \frac{-jk\exp(jkz)}{2\pi z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ds_x\, ds_y\, u_s(s_x,s_y)\exp\left\{\frac{jk}{2z}\left[-2s_xr_x-2s_yr_y+s_x^2+s_y^2+r_x^2+r_y^2\right]\right\} \quad \text{in Cartesian coordinates} \quad (1.2)$$

The presence of turbulence will be expressed an exponential term in the form of $\exp[\psi(\mathbf{r},\mathbf{s})]$, where $\psi(\mathbf{r},\mathbf{s})$ is the random part of the complex phase of a spherical wave propagating in the turbulent atmosphere from the source plane to the receiver plane at located at a distance z from the source. (1.1) and (1.2) are converted into extended Huygens – Fresnel integral as shown below

$$u_r(r,\phi_r,z)=\frac{-jk\exp(jkz)}{2\pi z}\int_0^\infty\int_0^{2\pi} ds d\phi_s s u_s(s,\phi_s)\exp\left\{\frac{jk}{2z}\left[-2rs\cos(\phi_r-\phi_s)+s^2+r^2\right]\right\}\exp[\psi(\mathbf{r},\mathbf{s})] \quad \text{in cylindrical coordinates} \quad (1.3)$$

$$u_r(r_x,r_y,z)=\frac{-jk\exp(jkz)}{2\pi z}\int_{-\infty}^\infty\int_{-\infty}^\infty ds_x ds_y u_s(s_x,s_y)\exp\left\{\frac{jk}{2z}\left[-2s_x r_x-2s_y r_y+s_x^2+s_y^2+r_x^2+r_y^2\right]\right\}\exp[\psi(\mathbf{r},\mathbf{s})] \quad \text{in Cartesian coordinates} \quad (1.4)$$

But we are rarely interested in the received field under atmospheric turbulence conditions. A more useful quantity in this case is the average intensity which is obtained from

$$\langle I(\mathbf{r},z=L)\rangle=\langle u_r(\mathbf{r},z=L)u_r^*(\mathbf{r},z=L)\rangle \quad \text{notation in either coordinates, where } \mathbf{r}=(r,\phi_r) \text{ or } \mathbf{r}=(r_x,r_y)$$

$$\langle I(r,\phi_r,z=L)\rangle=\langle u_r(r,\phi_r,z=L)u_r^*(r,\phi_r,z=L)\rangle \quad \text{in cylindrical coordinates}$$

$$\langle I(r_x,r_y,z=L)\rangle=\langle u_r(r_x,r_y,z=L)u_r^*(r_x,r_y,z=L)\rangle \quad \text{in Cartesian coordinates} \quad (1.5)$$

where $\langle\ \rangle$ and * refer to averaging and conjugate operations. It is clear from (1.3) and (1.4) that averaging will be applicable only to the random quantity $\exp[\psi(\mathbf{r},\mathbf{s})]$, since the other terms of the integrand are deterministic. This way, the average intensity on the receiver plane after a source beam of $u_s(\mathbf{s})$ propagates in turbulent atmosphere will become

$$\langle I(r,\phi_r,L)\rangle=\left(\frac{k}{2\pi L}\right)^2\int_0^\infty\int_0^{2\pi}\int_0^\infty\int_0^{2\pi} ds_1 d\phi_{s_1} ds_2 d\phi_{s_2} s_1 s_2 u_s(s_1,\phi_{s_1})\, u_s^*(s_2,\phi_{s_2})\exp\left\{\frac{jk}{2L}\left[-2rs_1\cos(\phi_r-\phi_{s_1})+s_1{}^2+2rs_2\cos(\phi_r-\phi_{s_2})-s_2{}^2\right]\right\}$$
$$\left\langle\exp\left[\psi(s_1,\phi_{s_1})+\psi^*(s_2,\phi_{s_2})\right]\right\rangle \quad \text{in cylindrical coordinates} \quad (1.6)$$

$$\left\langle I\left(r_x,r_y,L\right)\right\rangle=\left(\frac{k}{2\pi L}\right)^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}ds_{1x}ds_{1y}ds_{2x}ds_{2y}u_s\left(s_{1x},s_{1y}\right)u_s^*\left(s_{2x},s_{2y}\right)\exp\left[\frac{jk}{2L}\left(s_{1x}^2-2r_xs_{1x}+s_{1y}^2-2r_ys_{1y}-s_{2x}^2+2r_xs_{2x}-s_{2y}^2+2r_ys_{2y}\right)\right]$$
$$\left\langle\exp\left[\psi\left(s_{1x},s_{1y}\right)+\psi^*\left(s_{2x},s_{2y}\right)\right]\right\rangle \quad \text{in Cartesian coordinates} \quad (1.7)$$

Note that since in intensity, the receiver plane coordinate difference is zero, $\exp\left[\psi(\mathbf{r},\mathbf{s})\right]$ term turns into $\exp\left[\psi(\mathbf{s})\right]$.

$\psi(\mathbf{r}_1,\mathbf{s}_1)+\psi^*(\mathbf{r}_2,\mathbf{s}_2)$ is known as the wave structure function and for spherical wave, we use the following

$$\left\langle\exp\left[\psi\left(s_1,\phi_{s_1}\right)+\psi^*\left(s_2,\phi_{s_2}\right)\right]\right\rangle=\exp\left(-\frac{\left|\mathbf{s}_1-\mathbf{s}_2\right|^2}{\rho_t^2}\right)=\exp\left[-\frac{s_1^2+s_2^2-2s_1s_2\cos\left(\phi_{s_1}-\phi_{s_2}\right)}{\rho_t^2}\right] \quad \text{in cylindrical coordinates}$$

$$\left\langle\exp\left[\psi\left(s_{1x},s_{1y}\right)+\psi^*\left(s_{2x},s_{2y}\right)\right]\right\rangle=\exp\left(-\frac{\left|\mathbf{s}_1-\mathbf{s}_2\right|^2}{\rho_t^2}\right)=\exp\left(-\frac{s_{1x}^2+s_{2x}^2+s_{1y}^2+s_{2y}^2-2s_{1x}s_{2x}-2s_{1y}s_{2y}}{\rho_t^2}\right) \quad \text{in Cartesian coordinates} \quad (1.8)$$

where ρ_t is the spatial coherence length and is given by $\rho_t=\left(0.545C_n^2k^2L\right)^{-5/3}$ with C_n^2 is known the structure constant, denoting the turbulence level.

By inserting the expressions from (1.8) into (1.6) and (1.7), we can obtain the average receiver intensities of different beams propagating in turbulent atmosphere. Note that in both cylindrical and Cartesian coordinates, it is sufficient to perform only the double integration for the terms of indexed as 1, then the remaining double integration will be symmetrical.

Below we perform a sample derivation for a Sinusoidal Hyperbolic Gaussian beam.

2. The average intensity expression for partially coherent Sinusoidal Hyperbolic Gaussian beam propagating in turbulence

From the notes of ECE 635, we write the source plane field for this beam in Cartesian coordinates as

$$u_s\left(s_x, s_y\right)=\sum_{\ell=1}^{N} A_\ell \exp\left[-\left(0.5k\alpha_{x\ell}s_x^2-D_{x\ell}s_x\right)\right]\exp\left[-\left(0.5k\alpha_{y\ell}s_y^2-D_{y\ell}s_y\right)\right] \qquad (2.1)$$

where A_ℓ is the amplitude coefficient, $\alpha = 1/(k\alpha_s^2)+0.5j/F_s$ with α_s and F_s respectively referring to Gaussian source size and focusing parameter, D is the displacement parameter. Considering the form of partial coherence exponential in the derivation of mutual coherence function on pp. 14-17 of Notes for ECE 635, is similar to the turbulence exponential, we may benefit from the derivation of Notes for ECE 635 by establishing the following equivalence,

$$\underbrace{\exp\left(-\frac{s_{1x}^2+s_{2x}^2+s_{1y}^2+s_{2y}^2-2s_{1x}s_{2x}-2s_{1y}s_{2y}}{\rho_t^2}\right)}_{\text{Turbulence exponential}}\underbrace{\exp\left(-\frac{s_{1x}^2+s_{2x}^2+s_{1y}^2+s_{2y}^2-2s_{1x}s_{2x}-2s_{1y}s_{2y}}{2\sigma_s^2}\right)}_{\text{Partial coherence exponential}}$$

$$=\exp\left[-\left(\frac{2\sigma_s^2+\rho_t^2}{2\sigma_s^2\rho_t^2}\right)\left(s_{1x}^2+s_{2x}^2+s_{1y}^2+s_{2y}^2-2s_{1x}s_{2x}-2s_{1y}s_{2y}\right)\right] \qquad (2.2)$$

The arrangement in (2.2) implies that in (5.17) of ECE 635_Free space propagation notes_Eylul 2011_HTE, we need to replace all occurrences of σ_s^2 by $\frac{\sigma_s^2\rho_t^2}{2\sigma_s^2+\rho_t^2}$ in addition to the settings of $r_{1x}=r_{2x}=r_x$, $r_{1y}=r_{2y}=r_y$. Under these circumstances, the average intensity on the receiver plane of a partially coherent Sinusoidal Hyperbolic Gaussian beam after propagating in turbulence will become

$$\left\langle I\left(r_x,r_y,L\right)\right\rangle = k\sigma_s^2\rho_t^2\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\frac{A_{\ell_1}}{\left[k\alpha_{x\ell_1}\alpha_{x\ell_2}^*\sigma_s^2\rho_t^2L^2+\left(\alpha_{x\ell_1}+\alpha_{x\ell_2}^*\right)\left(2\sigma_s^2+\rho_t^2\right)L^2+j\left(\alpha_{x\ell_1}-\alpha_{x\ell_2}^*\right)k\sigma_s^2\rho_t^2L+k\sigma_s^2\rho_t^2\right]^{0.5}}$$

$$\times\frac{A_{\ell_1}^*}{\left[k\alpha_{y\ell_1}\alpha_{y\ell_2}^*\sigma_s^2\rho_t^2L^2+\left(\alpha_{y\ell_1}+\alpha_{y\ell_2}^*\right)\left(2\sigma_s^2+\rho_t^2\right)L^2+j\left(\alpha_{y\ell_1}-\alpha_{y\ell_2}^*\right)k\sigma_s^2\rho_t^2L+k\sigma_s^2\rho_t^2\right]^{0.5}}$$

$$\times\exp\left[\frac{0.5}{L\left[k\alpha_{x\ell_1}\sigma_s^2\rho_t^2L+\left(2\sigma_s^2+\rho_t^2\right)L-jk\sigma_s^2\rho_t^2\right]}\left(\left(-jkr_x+D_{x\ell_1}L\right)^2\sigma_s^2\rho_t^2\right.\right.$$

$$\left.\left.+\frac{\left\{j\left(\alpha_{x\ell_1}L-j\right)k^2r_x\sigma_s^2\rho_t^2+\left(2\sigma_s^2+\rho_t^2\right)D_{x\ell_1}L^2+\left[k\alpha_{x\ell_1}\sigma_s^2\rho_t^2L+\left(2\sigma_s^2+\rho_t^2\right)L-jk\sigma_s^2\rho_t^2\right]D_{x\ell_2}^*L\right\}^2}{k\left[k\alpha_{x\ell_1}\alpha_{x\ell_2}^*\sigma_s^2\rho_t^2L^2+\left(\alpha_{x\ell_1}+\alpha_{x\ell_2}^*\right)\left(2\sigma_s^2+\rho_t^2\right)L^2+j\left(\alpha_{x\ell_1}-\alpha_{x\ell_2}^*\right)k\sigma_s^2\rho_t^2L+k\sigma_s^2\rho_t^2\right]}\right)\right]$$

$$\times\exp\left[\frac{0.5}{L\left[k\alpha_{y\ell_1}\sigma_s^2\rho_t^2L+\left(2\sigma_s^2+\rho_t^2\right)L-jk\sigma_s^2\rho_t^2\right]}\left(\left(-jkr_y+D_{y\ell_1}L\right)^2\sigma_s^2\rho_t^2\right.\right.$$

$$\left.\left.+\frac{\left\{j\left(\alpha_{y\ell_1}L-j\right)k^2r_y\sigma_s^2\rho_t^2+\left(2\sigma_s^2+\rho_t^2\right)D_{y\ell_1}L^2+\left[k\alpha_{y\ell_1}\sigma_s^2\rho_t^2L+\left(2\sigma_s^2+\rho_t^2\right)L-jk\sigma_s^2\rho_t^2\right]D_{y\ell_2}^*L\right\}^2}{k\left[k\alpha_{y\ell_1}\alpha_{y\ell_2}^*\sigma_s^2\rho_t^2L^2+\left(\alpha_{y\ell_1}+\alpha_{y\ell_2}^*\right)\left(2\sigma_s^2+\rho_t^2\right)L^2+j\left(\alpha_{y\ell_1}-\alpha_{y\ell_2}^*\right)k\sigma_s^2\rho_t^2L+k\sigma_s^2\rho_t^2\right]}\right)\right] \quad (2.3)$$

Exercise 2.1 : Use ParCoh_SinoHypR_tur.m MATLAB file to plot the average intensity profiles of cos, cosh, sine, sinh and annular

Gaussian beams. Take measurements on these profiles and compare those measurements with those obtained from ParCoh_SinoHypR.m $C_n^2 \to 0$, which means free space limit, for at least five source and propagation settings. Also include in your report how intensity profiles of these beams change when $C_n^2 = 10^{-15}$ m$^{-2/3}$, $C_n^2 = 10^{-14}$ m$^{-2/3}$, $C_n^2 = 10^{-13}$ m$^{-2/3}$.

3. Rytov Scintillation Theory

Born Approximation

For a scalar field U propagating in an turbulent atmosphere (characterized as random medium) whose refractive index is given by $n(\mathbf{R})$, the Helmholtz equation is

$$\nabla^2 U + k^2 n^2(\mathbf{R}) U = 0 \quad , \ \mathbf{R} = (r_x, r_y, z) \text{ or } \mathbf{R} = (r, \phi_r, z) \qquad (3.1)$$

where $n(\mathbf{R}) = n_0 + n_1(\mathbf{R}) \qquad (3.2)$

such that $n_0 = \langle n(\mathbf{R}) \rangle \cong 1$, $\langle n_1(\mathbf{R}) \rangle = 0$. Furthermore $n^2(\mathbf{R}) = [n_0 + n_1(\mathbf{R})]^2 \cong 1 + 2n_1(\mathbf{R})$ since $|n_1(\mathbf{R})| \ll 1$. To solve (1.1) using Born approximation, we assume that we can expand $U(\mathbf{R})$ as follows (note that we use $U(\mathbf{R})$ and U synonymously)

$$U(\mathbf{R}) = U_0(\mathbf{R}) + U_1(\mathbf{R}) + U_2(\mathbf{R}) + \cdots \qquad (3.3)$$

This way $U_0(\mathbf{R})$ is the unperturbed free space field, $U_1(\mathbf{R})$ is the perturbed field due to first order scattering (caused by turbulence), $U_2(\mathbf{R})$ due to second order and so on. The usual assumption is that $|U_0(\mathbf{R})| \gg |U_1(\mathbf{R})| \gg |U_2(\mathbf{R})|$. Substituting (3.3) in (3.1) and using the approximation of the refractive index, thereby equating the same ordered terms (bear in mind that $n_1(\mathbf{R})$ adds order of one), we get

$$\begin{aligned} &\nabla^2 U_0 + k^2 U_0 = 0 \\ &\nabla^2 U_1 + k^2 U_1 = -2k^2 n_1(\mathbf{R}) U_0(\mathbf{R}) \\ &\nabla^2 U_2 + k^2 U_2 = -2k^2 n_1(\mathbf{R}) U_1(\mathbf{R}) \end{aligned} \tag{3.4}$$

(3.4) means once U_0 is known, then it is possible to determine higher order $U_1(\mathbf{R})$, $U_2(\mathbf{R})$. To obtain $U_1(\mathbf{R})$ form $U_0(\mathbf{R})$, we use the following integral

$$U_1(\mathbf{R}) = \iiint_V G(\mathbf{S},\mathbf{R}) \left[2k^2 n_1(\mathbf{R}) U_0(\mathbf{R}) \right] d^3 S \tag{3.5}$$

where $G(\mathbf{S},\mathbf{R})$ is the free space Green's function defined as

$$G(\mathbf{S},\mathbf{R}) = \frac{1}{4\pi |\mathbf{R}-\mathbf{S}|} \exp\left(jk |\mathbf{R}-\mathbf{S}| \right) \tag{3.6}$$

By noting that the changes in the transverse distances are much less than those in the longitudinal distances, thus we can employ paraxial approximation, then with notation of $\mathbf{R} = (\mathbf{r}, L)$, $\mathbf{S} = (\mathbf{s}, z)$ and the Green's function will become $G(\mathbf{S},\mathbf{R})$ (here $\mathbf{s}$ is used as a dummy receiver coordinate and should not be confused with source coordinate $\mathbf{s}$)

$$G(\mathbf{S},\mathbf{R}) \cong \frac{1}{4\pi(L-z)} \exp\left[jk(L-z) + \frac{jk|\mathbf{s}-\mathbf{r}|^2}{2(L-z)}\right] \tag{3.7}$$

Inserting (3.7) into (3.5), we get

$$U_1(\mathbf{R}) = \frac{k^2}{2\pi} \int_0^L dz \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2 s \exp\left[jk(L-z) + \frac{jk|\mathbf{s}-\mathbf{r}|^2}{2(L-z)}\right] \frac{U_0(\mathbf{s},z)}{L-z}\, n_1(\mathbf{s},z) \tag{3.8}$$

For a general order *m*, (3.8) will simply turn into

$$U_m(\mathbf{R}) = \frac{k^2}{2\pi} \int_0^L dz \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2 s \exp\left[jk(L-z) + \frac{jk|\mathbf{s}-\mathbf{r}|^2}{2(L-z)}\right] \frac{U_{m-1}(\mathbf{s},z)}{L-z}\, n_1(\mathbf{s},z) \tag{3.9}$$

It is important to point out that Born approximation is valid only over extremely short propagation distances. But above formulation will be useful for Rytov method described below.

<u>Rytov Approximation</u>

In Rytov method, the main distinction is that perturbations due to randomness of the propagation medium are represented by an exponential complex phase as shown below

$$U(\mathbf{R}) = U(\mathbf{r},L) = U_0(\mathbf{r},L)\exp[\psi(\mathbf{r},L)] \tag{3.10}$$

Here complex phase $\psi(\mathbf{r},L)$ can be expanded as

$$\psi(\mathbf{r},L) = \psi_1(\mathbf{r},L) + \psi_2(\mathbf{r},L) + \psi_3(\mathbf{r},L) + \psi_4(\mathbf{r},L) + \cdots \tag{3.11}$$

The numerically indexed terms can be termed as first and second order perturbations. We can apply (3.10) to (3.1) to arrive at Rytov solutions. But it is much simpler to obtain those from the already developed Born approximation. To this end we introduce the normalized Born pertubation defined as

$$\Phi_m(\mathbf{r},L)=\frac{U_m(\mathbf{r},L)}{U_0(\mathbf{r},L)} \qquad m=1,2,3,..... \tag{3.12}$$

Now we equate first order Rytov and Born perturbations such that

$$U_0(\mathbf{r},L)\exp[\psi_1(\mathbf{r},L)]=U_0(\mathbf{r},L)+U_1(\mathbf{r},L)=U_0(\mathbf{r},L)[1+\Phi_1(\mathbf{r},L)] \tag{3.13}$$

From (3.13), we find $\psi_1(\mathbf{r},L)$ to be

$$\psi_1(\mathbf{r},L)=\ln[1+\Phi_1(\mathbf{r},L)]\cong\Phi_1(\mathbf{r},L) \qquad \text{since } |\Phi_1(\mathbf{r},L)|\ll 1 \tag{3.14}$$

Thus $\psi_1(\mathbf{r},L)$ will become

$$\psi_1(\mathbf{r},L)\cong\Phi_1(\mathbf{r},L)=\frac{U_1(\mathbf{r},L)}{U_0(\mathbf{r},L)}=\frac{k^2}{2\pi U_0(\mathbf{r},L)}\int_0^L dz\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2s\exp\left[jk(L-z)+\frac{jk|\mathbf{s}-\mathbf{r}|^2}{2(L-z)}\right]\frac{U_0(\mathbf{s},z)n_1(\mathbf{s},z)}{(L-z)} \tag{3.15}$$

By equating Rytov and Born perturbations up to the second order, we find that

$$\begin{aligned}\psi_1(\mathbf{r},L)+\psi_2(\mathbf{r},L)&=\ln[1+\Phi_1(\mathbf{r},L)+\Phi_2(\mathbf{r},L)]\\ &\cong\Phi_1(\mathbf{r},L)+\Phi_2(\mathbf{r},L)-0.5\Phi_1^2(\mathbf{r},L) \qquad \text{since } |\Phi_1(\mathbf{r},L)|\ll 1 \text{ and } |\Phi_2(\mathbf{r},L)|\ll|\Phi_1(\mathbf{r},L)|\end{aligned} \tag{3.16}$$

Using (3.14), $\psi_2(\mathbf{r},L)$ will be

$$\psi_2(\mathbf{r},L) = \Phi_2(\mathbf{r},L) - 0.5\Phi_1^2(\mathbf{r},L) \qquad (3.17)$$

where $\Phi_2(\mathbf{r},L)$ can be written as

$$\Phi_2(\mathbf{r},L) = \frac{U_2(\mathbf{r},L)}{U_0(\mathbf{r},L)} = \frac{k^2}{2\pi U_0(\mathbf{r},L)} \int_0^L dz \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2 s \exp\left[jk(L-z) + \frac{jk|\mathbf{s}-\mathbf{r}|^2}{2(L-z)} \right] \frac{U_0(\mathbf{s},z)\Phi_1(\mathbf{s},z) n_1(\mathbf{s},z)}{(L-z)} \qquad (3.18)$$

As we shall see below, for scintillation the use of $\psi_1(\mathbf{r},L)$ will be sufficient, whereas the calculation of $\langle U(\mathbf{r},L)\rangle$ will require second order perturbation.

Now covering the first and second order perturbations we need the following (ensemble) averages for the field, intensity and intensity square

$\langle U(\mathbf{r},L)\rangle$ requires the calulation of $\langle \exp[\psi(\mathbf{r},L)]\rangle = \langle \exp[\psi_1(\mathbf{r},L) + \psi_2(\mathbf{r},L)]\rangle$

$\langle I(\mathbf{r},L)\rangle$ requires the calulation of $\langle \exp[\psi(\mathbf{r}_1,L) + \psi^*(\mathbf{r}_2,L)]\rangle = \langle \exp[\psi_1(\mathbf{r}_1,L) + \psi_2(\mathbf{r}_1,L) + \psi_1^*(\mathbf{r}_2,L) + \psi_2^*(\mathbf{r}_2,L)]\rangle$

$\langle I^2(\mathbf{r},L)\rangle$ requires the calulation of $\langle \exp[\psi(\mathbf{r}_1,L) + \psi^*(\mathbf{r}_2,L) + \psi(\mathbf{r}_3,L) + \psi^*(\mathbf{r}_4,L)]\rangle$

$$= \langle \exp[\psi_1(\mathbf{r}_1,L) + \psi_2(\mathbf{r}_1,L) + \psi_1^*(\mathbf{r}_2,L) + \psi_2^*(\mathbf{r}_2,L) + \psi_1(\mathbf{r}_3,L) + \psi_2(\mathbf{r}_3,L) + \psi_1^*(\mathbf{r}_4,L) + \psi_2^*(\mathbf{r}_4,L)]\rangle \qquad (3.19)$$

By using the following second order approximation

$\langle \exp(\psi)\rangle = \exp\left[\langle\psi\rangle + 0.5\left(\langle\psi^2\rangle - \langle\psi\rangle^2\right)\right]$ from (14) of Andrews 2005 on pp.184 (3.20) and noting that $\langle n_1(\mathbf{R})\rangle = 0$ (as indicated above), thus $\langle \psi_1(\mathbf{r},L)\rangle = 0$ due to (3.15), then we find for the ensemble averages of the exp expressions in (3.19)

$$\left\langle \exp\left[\psi_1(\mathbf{r},L)+\psi_2(\mathbf{r},L)\right]\right\rangle = \exp\left[E_1(0,0)\right]$$
$$\left\langle \exp\left[\psi_1(\mathbf{r}_1,L)+\psi_2(\mathbf{r}_1,L)+\psi_1^*(\mathbf{r}_2,L)+\psi_2^*(\mathbf{r}_2,L)\right]\right\rangle = \exp\left[2E_1(0,0)+E_2(\mathbf{r}_1,\mathbf{r}_2)\right]$$
$$\left\langle \exp\left[\psi_1(\mathbf{r}_1,L)+\psi_2(\mathbf{r}_1,L)+\psi_1^*(\mathbf{r}_2,L)+\psi_2^*(\mathbf{r}_2,L)+\psi_1(\mathbf{r}_3,L)+\psi_2(\mathbf{r}_3,L)+\psi_1^*(\mathbf{r}_4,L)+\psi_2^*(\mathbf{r}_4,L)\right]\right\rangle$$
$$= \exp\left[4E_1(0,0)+E_2(\mathbf{r}_1,\mathbf{r}_2)+E_2(\mathbf{r}_1,\mathbf{r}_4)+E_2(\mathbf{r}_2,\mathbf{r}_3)+E_2(\mathbf{r}_3,\mathbf{r}_4)+E_3(\mathbf{r}_1,\mathbf{r}_3)+E_3^*(\mathbf{r}_2,\mathbf{r}_4)\right] \quad (3.21)$$

where, E_1, E_2 and E_3 in terms of ψ_1 and ψ_2 are

$$E_1(0,0) = \langle \psi_2(\mathbf{r},L)\rangle + 0.5\langle \psi_1^2(\mathbf{r},L)\rangle$$
$$E_2(\mathbf{r}_1,\mathbf{r}_2) = \langle \psi_1(\mathbf{r}_1,L)\psi_1^*(\mathbf{r}_2,L)\rangle$$
$$E_3(\mathbf{r}_1,\mathbf{r}_2) = \langle \psi_1(\mathbf{r}_1,L)\psi_1(\mathbf{r}_2,L)\rangle \quad (3.22)$$

By using the definitions given in (3.15), (3.17) and (3.18), it is possible to evaluate E_1, E_2 and E_3. Despite the awkward appearance however, E_1 comes out the simplest, source beam independent and contains only spectrum function dependence.

Scintillation index as a measure of normalized variance of amplitude fluctuations in the beam that has traversed a turbulent medium is given by

$$m^2(\mathbf{r},L) = \frac{\langle I^2(\mathbf{r},L)\rangle - \langle I(\mathbf{r},L)\rangle^2}{\langle I(\mathbf{r},L)\rangle^2} = \frac{\langle I^2(\mathbf{r},L)\rangle}{\langle I(\mathbf{r},L)\rangle^2} - 1 \quad (3.23)$$

Here the numerator (of the first expression) refers to the statistical definition of variance, while the $\left\langle I\left(\mathbf{r},L\right)\right\rangle^2$ serves for normalization. It is possible to acquire quantities $\left\langle I^2\left(\mathbf{r},L\right)\right\rangle$ and $\left\langle I\left(\mathbf{r},L\right)\right\rangle^2$, that is average intensity square and squared average intensity by defining mutual coherence function Γ related to the above developments. This way

$$\begin{aligned}\Gamma\left(\mathbf{r_1},\mathbf{r_2},L\right)&=\left\langle U\left(\mathbf{r_1},L\right)U^*\left(\mathbf{r_2},L\right)\right\rangle=U_0\left(\mathbf{r_1},L\right)U_0^*\left(\mathbf{r_2},L\right)\left\langle \exp\left[\psi\left(\mathbf{r_1},L\right)+\psi^*\left(\mathbf{r_2},L\right)\right]\right\rangle\\
&\cong U_0\left(\mathbf{r_1},L\right)U_0^*\left(\mathbf{r_2},L\right)\left\langle \exp\left[\psi_1\left(\mathbf{r}_1,L\right)+\psi_2\left(\mathbf{r}_1,L\right)+\psi_1^*\left(\mathbf{r}_2,L\right)+\psi_2^*\left(\mathbf{r}_2,L\right)\right]\right\rangle\\
&=U_0\left(\mathbf{r_1},L\right)U_0^*\left(\mathbf{r_2},L\right)\exp\left[2E_1\left(0,0\right)+E_2\left(\mathbf{r_1},\mathbf{r_2}\right)\right]\end{aligned}\tag{3.24}$$

$$\begin{aligned}\Gamma\left(\mathbf{r_1},\mathbf{r_2},\mathbf{r_3},\mathbf{r_4},L\right)&=\left\langle U\left(\mathbf{r_1},L\right)U^*\left(\mathbf{r_2},L\right)U\left(\mathbf{r_3},L\right)U^*\left(\mathbf{r_4},L\right)\right\rangle\\
&=U_0\left(\mathbf{r_1},L\right)U_0^*\left(\mathbf{r_2},L\right)U_0\left(\mathbf{r_3},L\right)U_0^*\left(\mathbf{r_4},L\right)\left\langle \exp\left[\psi\left(\mathbf{r_1},L\right)+\psi^*\left(\mathbf{r_2},L\right)+\psi\left(\mathbf{r_3},L\right)+\psi^*\left(\mathbf{r_4},L\right)\right]\right\rangle\\
&=U_0\left(\mathbf{r_1},L\right)U_0^*\left(\mathbf{r_2},L\right)U_0\left(\mathbf{r_3},L\right)U_0^*\left(\mathbf{r_4},L\right)\\
&\times\exp\left[4E_1\left(0,0\right)+E_2\left(\mathbf{r_1},\mathbf{r_2}\right)+E_2\left(\mathbf{r_1},\mathbf{r_4}\right)+E_2\left(\mathbf{r_2},\mathbf{r_3}\right)+E_2\left(\mathbf{r_3},\mathbf{r_4}\right)+E_3\left(\mathbf{r_1},\mathbf{r_3}\right)+E_3^*\left(\mathbf{r_2},\mathbf{r_4}\right)\right]\\
&=\Gamma\left(\mathbf{r_1},\mathbf{r_2},L\right)\Gamma\left(\mathbf{r_3},\mathbf{r_4},L\right)\exp\left[E_2\left(\mathbf{r_1},\mathbf{r_4}\right)+E_2\left(\mathbf{r_2},\mathbf{r_3}\right)+E_3\left(\mathbf{r_1},\mathbf{r_3}\right)+E_3^*\left(\mathbf{r_2},\mathbf{r_4}\right)\right]\end{aligned}\tag{3.25}$$

We know that intensity is equal to the mutual coherence function whose transverse coordinates ($\mathbf{r}$s) are made coincident. This means that

$$\langle I(\mathbf{r},L)\rangle^2 = \Gamma^2(\mathbf{r_1}=\mathbf{r},\mathbf{r_2}=\mathbf{r},L) = U_0^2(\mathbf{r_1}=\mathbf{r},L)\left[U_0^*(\mathbf{r_2}=\mathbf{r},L)\right]^2 \exp\left[4E_1(0,0)+2E_2(\mathbf{r_1}=\mathbf{r},\mathbf{r_2}=\mathbf{r})\right] = U_0^2(\mathbf{r},L)\left[U_0^*(\mathbf{r},L)\right]^2 \exp\left[4E_1(0,0)+2E_2(\mathbf{r},\mathbf{r})\right]$$

$$\begin{aligned}\langle I^2(\mathbf{r},L)\rangle &= \Gamma(\mathbf{r_1}=\mathbf{r},\mathbf{r_2}=\mathbf{r},\mathbf{r_3}=\mathbf{r},\mathbf{r_4}=\mathbf{r},L) = U_0(\mathbf{r_1}=\mathbf{r},L)U_0^*(\mathbf{r_2}=\mathbf{r},L)U_0(\mathbf{r_3}=\mathbf{r},L)U_0^*(\mathbf{r_4}=\mathbf{r},L) \\ &\times \exp\left[4E_1(0,0)+E_2(\mathbf{r_1},\mathbf{r_2})+E_2(\mathbf{r_1},\mathbf{r_4})+E_2(\mathbf{r_2},\mathbf{r_3})+E_2(\mathbf{r_3},\mathbf{r_4})+E_3(\mathbf{r_1},\mathbf{r_3})+E_3^*(\mathbf{r_2},\mathbf{r_4})\right] \\ &= U_0^2(\mathbf{r},L)\left[U_0^*(L)\right]^2 \exp\left[4E_1(0,0)+E_2(\mathbf{r},\mathbf{r})+E_2(\mathbf{r},\mathbf{r})+E_2(\mathbf{r},\mathbf{r})+E_2(\mathbf{r},\mathbf{r})+E_3(\mathbf{r},\mathbf{r})+E_3^*(\mathbf{r},\mathbf{r})\right] \\ &= \langle I(\mathbf{r},L)\rangle^2 \exp\left\{2E_2(\mathbf{r},\mathbf{r})+2\,\mathrm{Re}\left[E_3(\mathbf{r},\mathbf{r})\right]\right\} \end{aligned} \qquad (3.26)$$

Substituting (3.26) in (3.23) will deliver

$$\begin{aligned} m^2(\mathbf{r},L) &= \frac{\langle I(\mathbf{r},L)\rangle^2 \exp\left\{2E_2(\mathbf{r},\mathbf{r})+2\,\mathrm{Re}\left[E_3(\mathbf{r},\mathbf{r})\right]\right\}}{\langle I(\mathbf{r},L)\rangle^2} - 1 \\ &= \exp\left\{2E_2(\mathbf{r},\mathbf{r})+2\,\mathrm{Re}\left[E_3(\mathbf{r},\mathbf{r})\right]\right\} - 1 \cong 2E_2(\mathbf{r},\mathbf{r})+2\,\mathrm{Re}\left[E_3(\mathbf{r},\mathbf{r})\right] = 2\langle \psi_1(\mathbf{r},L)\psi_1^*(\mathbf{r},L)\rangle + 2\,\mathrm{Re}\left[\langle \psi_1(\mathbf{r},L)\psi_1(\mathbf{r},L)\rangle\right] \end{aligned} \qquad (3.27)$$

The approximation on the second line of (3.27) is due to $2E_2(\mathbf{r},\mathbf{r})+2\,\mathrm{Re}\left[E_3(\mathbf{r},\mathbf{r})\right]$ being too small under weak turbulence conditions. (3.27) shows that calculation of scintillation index is reduced to the evaluation of a single function, namely $\psi_1(\mathbf{r},L)$ or rather the ensemble averages of $\psi_1(\mathbf{r},L)$ multiplied by its conjugate and by itself (i.e. E_2 and E_3).

If we consider (3.15), we come to the conclusion that the ensemble operation to be applied to $\psi_1(\mathbf{r},L)$ will entail only $n_1(\mathbf{s},z)$, since all other terms in (3.15) are deterministic. Thus

$$\left\langle \psi_1(\mathbf{r},L)\psi_1^*(\mathbf{r},L)\right\rangle = \frac{k^4}{4\pi^2\left|U_0(\mathbf{r},L)\right|^2}\int_0^L dz\int_0^L dz_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2 s \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2 s_1 \exp\left[jk(L-z) - jk(L-z_1) + \frac{jk\left|\mathbf{s}-\mathbf{r}\right|^2}{2(L-z)} - \frac{jk\left|\mathbf{s}_1-\mathbf{r}\right|^2}{2(L-z_1)}\right]$$
$$\times \frac{U_0(\mathbf{s},z)U_0^*(\mathbf{s}_1,z_1)}{(L-z)(L-z_1)}\left\langle n_1(\mathbf{s},z)n_1^*(\mathbf{s}_1,z_1)\right\rangle \qquad (3.28)$$

Now we take out $\left\langle n_1(\mathbf{s},z)n_1(\mathbf{s}_1,z_1)\right\rangle$ and consider this term together with distance integrals. From pp. 145 of Andrews 2005, we know that refractive index fluctuations n_1 can be written as two dimensional Reimann-Stieltjets integral, then

$$\int_0^L dz\int_0^L dz_1 \left\langle n_1(\mathbf{s},z)n_1^*(\mathbf{s}_1,z_1)\right\rangle = \int_0^L dz\int_0^L dz_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(j \quad \cdot \quad)\exp(-j \quad_1\cdot \quad_1)\left\langle d\nu(\quad,z)d\nu^*(\quad_1,z_1)\right\rangle \qquad (3.29)$$

Here $\left\langle d\nu(\boldsymbol{\kappa},z)d\nu^*(\boldsymbol{\kappa}_1,z_1)\right\rangle$ is related to spectral density function F_n whose in turn is related to spatial power spectrum function Φ_n via Fourier Transform relationship, thus

$$\left\langle d\nu(\boldsymbol{\kappa},z)d\nu^*(\boldsymbol{\kappa}_1,z_1)\right\rangle = F_n(\boldsymbol{\kappa},|z-z_1|)\delta(\boldsymbol{\kappa}-\boldsymbol{\kappa}_1)d^2\kappa d^2\kappa_1 \qquad (3.30)$$

Substituting (3.30) into (3.29) and performing the double integration over κ_1 taking into account the delta function in (3.30) will give

$$\int_0^L dz\int_0^L dz_1 \left\langle n_1(\mathbf{s},z)n_1^*(\mathbf{s}_1,z_1)\right\rangle = \int_0^L dz\int_0^L dz_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\kappa \exp(j \quad \cdot \quad)\exp(-j \quad \cdot \quad_1)F_n(\quad,|z-z_1|) \qquad (3.31)$$

As explained on pp. 148 and 149 of Andrews 2005, we benefit from the fact that F_n has its spectrum mostly confined to the region around $z \cong z_1$. This way we can make a change of distance variables to sum and difference of z and z_1 and extend the range of integration over the difference to minus and plus infinity so that we can replace the density function F_n by the spectrum function Φ_n. Hence eventually we get

$$\int_0^L dz \int_0^L dz_1 \left\langle n_1(\mathbf{s},z) n_1^*(\mathbf{s}_1,z_1) \right\rangle = 2\pi \int_0^L d\eta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2\kappa \exp(j \quad \cdot \quad) \exp(-j \quad \cdot \quad _1) \Phi_n(\quad) \tag{3.32}$$

Note that η is the new distance variable. We could have equally used z, so the change to η is merely to be in line with the notation of Scintillation Formulation Via Rytov_HTE_Nisan 2009 notes. By inserting the result found in (3.32) back into (3.28), leads to the following

$$\left\langle \psi_1(\mathbf{r},L) \psi_1^*(\mathbf{r},L) \right\rangle = \frac{2\pi k^4}{4\pi^2 \left|U_0(\mathbf{r},L)\right|^2} \int_0^L d\eta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2 s \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2 s_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2\kappa \exp\left[\frac{jk\left|\mathbf{s}-\mathbf{r}\right|^2}{2(L-\eta)} - \frac{jk\left|\mathbf{s}_1-\mathbf{r}\right|^2}{2(L-\eta)}\right] \frac{U_0(\mathbf{s},\eta) U_0^*(\mathbf{s}_1,\eta)}{(L-\eta)^2} \exp(j \quad \cdot \quad) \exp(-j \quad \cdot \quad _1) \Phi_n(\quad)$$

$$= \frac{2\pi k^4}{4\pi^2 \left|U_0(\mathbf{r},L)\right|^2} \int_0^L d\eta \int_0^\infty d\kappa \int_0^{2\pi} d\phi_\kappa \, \frac{\kappa \, \Phi_n(\kappa)}{(L-\eta)^2} \int_0^\infty \int_0^{2\pi} ds d\phi s U_0(s,\phi,\eta) \exp\left[j\kappa s \cos(\phi-\phi_\kappa)\right] \exp\left\{\frac{jk}{2(L-\eta)}\left[s^2 - 2sr\cos(\phi-\phi_r)\right]\right\}$$

$$\times \int_0^\infty \int_0^{2\pi} ds_1 d\phi_1 s_1 U_0^*(s_1,\phi_1,\eta) \exp\left[-j\kappa s_1 \cos(\phi_1-\phi_\kappa)\right] \exp\left\{\frac{-jk}{2(L-\eta)}\left[s_1^2 - 2s_1 r\cos(\phi_1-\phi_r)\right]\right\} = 2\pi \int_0^L d\eta \int_0^\infty \int_0^{2\pi} d^2\kappa\kappa \, \Phi_n(\kappa) \, H(r,\phi_r,\kappa,\phi_\kappa,\eta) H^*(r,\phi_r,\kappa,\phi_\kappa,\eta) \tag{3.33}$$

On the last line of (3.33), the equivalence to (7) of notes Scintillation Formulation Via Rytov_HTE_Nisan 2009 is established.

To get $\left\langle \psi_1(\mathbf{r},L) \psi_1(\mathbf{r},L) \right\rangle$ or E_2, we keep in mind that $n_1(\mathbf{s},z)$ is a real function, which means

$$\left\langle d\nu(\boldsymbol{\kappa},z) d\nu(\boldsymbol{\kappa}_1,z_1) \right\rangle = \left\langle d\nu(\boldsymbol{\kappa},z) d\nu^*(-\boldsymbol{\kappa}_1,z_1) \right\rangle = F_n\left(\boldsymbol{\kappa},\left|z-z_1\right|\right) \delta(\boldsymbol{\kappa}+\boldsymbol{\kappa}_1) d^2\kappa d^2\kappa_1 \tag{3.34}$$

So the above of $\langle \psi_1(\mathbf{r},L)\psi_1^*(\mathbf{r},L)\rangle$ can be applied to $\langle \psi_1(\mathbf{r},L)\psi_1(\mathbf{r},L)\rangle$ by changing all $\boldsymbol{\kappa}_1$ with $-\boldsymbol{\kappa}$, in this manner $\langle \psi_1(\mathbf{r},L)\psi_1(\mathbf{r},L)\rangle$ will be

$$\langle \psi_1(\mathbf{r},L)\psi_1(\mathbf{r},L)\rangle = \frac{2\pi k^4}{4\pi^2 U_0^2(\mathbf{r},L)}\int_0^L d\eta \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2 s \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2 s_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\kappa \exp\left[\frac{jk|\mathbf{s}-\mathbf{r}|^2}{2(L-\eta)}+\frac{jk|\mathbf{s}_1-\mathbf{r}|^2}{2(L-\eta)}\right]\frac{U_0(\mathbf{s},\eta)U_0(\mathbf{s}_1,\eta)}{(L-\eta)^2}\exp(j \ \cdot \ )\exp(-j \ \cdot \ _1)\Phi_n(\ )$$

$$= \frac{2\pi k^4}{4\pi^2 U_0^2(\mathbf{r},L)}\int_0^L d\eta \int_0^\infty d\kappa \int_0^{2\pi} d\phi_\kappa \, \frac{\kappa \, \Phi_n(\kappa)}{(L-\eta)^2}\exp\left(\frac{jkr^2}{L-\eta}\right)\int_0^\infty\int_0^{2\pi} ds d\phi s U_0(s,\phi,\eta)\exp\left[j\kappa s\cos(\phi-\phi_\kappa)\right]\exp\left\{\frac{jk}{2(L-\eta)}\left[s^2-2sr\cos(\phi-\phi_r)\right]\right\}$$

$$\times\int_0^\infty\int_0^{2\pi} ds_1 d\phi_1 s_1 U_0(s_1,\phi_1,\eta)\exp\left[-j\kappa s_1\cos(\phi_1-\phi_\kappa)\right]\exp\left\{\frac{jk}{2(L-\eta)}\left[s_1^2-2s_1 r\cos(\phi_1-\phi_r)\right]\right\} = 2\pi\int_0^L d\eta \int_0^\infty\int_0^{2\pi} d^2\kappa \, \kappa \, \Phi_n(\kappa) H(r,\phi_r,\kappa,\phi_\kappa,\eta) H(r,\phi_r,-\kappa,\phi_\kappa,\eta) \quad (3.35)$$

Finally we remind that in terms of scintillation index under weak fluctuation conditions will be given by

$$m^2(\mathbf{r},L) = 2E_2(\mathbf{r},\mathbf{r}) + 2\,\mathrm{Re}\left[E_3(\mathbf{r},\mathbf{r})\right] = 2\left\{\langle \psi_1(\mathbf{r},L)\psi_1^*(\mathbf{r},L)\rangle + \mathrm{Re}\left[\langle \psi_1(\mathbf{r},L)\psi_1(\mathbf{r},L)\rangle\right]\right\} \quad (3.36)$$

Below we give an application example of the above type derivation.

Sample scintillation derivation for Sinusoidal Hyperbolic Gaussian beam

We cite source field as

$$u_s(s,\phi_s) = \sum_{\ell=1}^{N} A_\ell \exp\left[-k\alpha_\ell s^2 + (\sin\phi_s + \cos\phi_s) D_{s\ell} s\right] \qquad \text{in cylindrical coordinates} \qquad (3.37)$$

The first step is to find $U_0(\mathbf{r},L)$, which is the free space receiver field. From Q1 of MT Exam of ECE 635 dated 22.11.2011, we have

$$U_0(\mathbf{r},L)=U_0(r,\phi_r,L)=\sum_{\ell=1}^{2}\frac{A_\ell}{1+2j\alpha_\ell L}\exp\left[-\frac{k\alpha_\ell r^2-(\cos\phi_r+\sin\phi_r)rD_{s\ell}-jk^{-1}D_{s\ell}^2L}{1+2j\alpha_\ell L}\right] \quad (3.38)$$

Now we apply the double integration over $\mathbf{s}$ in (3.15) including the $\boldsymbol{\kappa}$ exp from $n_1(\mathbf{s},z)$ or apply (7) from Scintillation Formulation Via Rytov_HTE_Nisan 2009 to obtain

$$H(r,\phi_r,\kappa,\phi_\kappa,\eta)=jk\sum_{\ell=1}^{2}\frac{A_\ell}{1+2j\alpha_\ell L}\exp\left[-\frac{k\alpha_\ell r^2}{1+2j\alpha_\ell L}-0.5j\frac{\kappa^2(1+2j\alpha_\ell\eta)(L-\eta)}{k(1+2j\alpha_\ell L)}+\frac{j\kappa r(1+2j\alpha_\ell\eta)}{1+2j\alpha_\ell L}\cos(\phi_\kappa-\phi_r)-\frac{\kappa D_{s\ell}(L-\eta)}{k(1+2j\alpha_\ell L)}(\cos\phi_\kappa+\sin\phi_\kappa)\right]$$
$$\times\exp\left[\frac{rD_{s\ell}}{1+2j\alpha_\ell L}(\cos\phi_r+\sin\phi_r)+j\frac{D_{s\ell}^2L}{k(1+2j\alpha_\ell L)}\right]\Bigg/\sum_{\ell=1}^{2}\frac{A_\ell}{1+2j\alpha_\ell L}\exp\left[-\frac{k\alpha_\ell r^2-(\cos\phi_r+\sin\phi_r)rD_{s\ell}-jk^{-1}D_{s\ell}^2L}{1+2j\alpha_\ell L}\right] \quad (3.39)$$

Lab Exercise : Verify (3.39) by using the following useful formulations

$$\int_0^{2\pi}dx\exp(jp\cos x+jq\sin x)=2\pi J_0\left(\sqrt{p^2+q^2}\right)$$ for integration over the dummy variable inserted for ϕ_r (3.40)

$$\int_0^{\infty}dx\,x^{\nu+1}\exp\left(-ax^2\right)J_\nu(\beta x)=\frac{\beta^\nu}{(2a)^{\nu+1}}\exp\left(-\frac{\beta^2}{4a}\right)$$ for integration over the dummy variable inserted for r (3.41)

Now we take the (modified) definition of $H(\ )$, which is

$$H(r,\phi_r,\kappa,\phi_\kappa,\eta)=\frac{k^2}{2\pi(L-\eta)U_0(r,\phi_r,z=L)}\exp\left[\frac{jkr^2}{2(L-\eta)}\right]\int_0^{\infty}\int_0^{2\pi}dr_1d\phi_1r_1U_0(r_1,\phi_1,z=\eta)\exp\left[j\kappa r_1\cos(\phi_1-\phi_\kappa)\right]\exp\left\{\frac{jk}{2(L-\eta)}\left[r_1^2-2r_1r\cos(\phi_1-\phi_r)\right]\right\} \quad (3.42)$$

After insering for $U_0(\ )$ into (3.42) from (3.38), the integration over ϕ_1 looks like

$$I_{\phi_1} = \int_0^{2\pi} d\phi_1 \exp\left[\left(j\kappa r_1 \cos\phi_\kappa - \frac{jk}{L-\eta} r r_1 \cos\phi_r + \frac{r_1 D_{s\ell}}{1+2j\alpha_\ell L}\right)\cos\phi_1 + \left(j\kappa r_1 \sin\phi_\kappa - \frac{jk}{L-\eta} r r_1 \sin\phi_r + \frac{r_1 D_{s\ell}}{1+2j\alpha_\ell L}\right)\sin\phi_1\right] \quad (3.43)$$

By making an association between (3.43) and (3.40), we get p and q as (note that we exclude r_1, since it is common to all terms)

$$p = \kappa\cos\phi_\kappa - \frac{k}{L-\eta} r\cos\phi_r - \frac{jD_{s\ell}}{1+2j\alpha_\ell L} \quad , \quad q = \kappa\sin\phi_\kappa - \frac{k}{L-\eta} r\sin\phi_r - \frac{jD_{s\ell}}{1+2j\alpha_\ell L} \quad (3.44)$$

So the result of (3.43) will be $I_{\phi_1} = 2\pi J_0\left(r_1\sqrt{p^2+q^2}\right)$, where p and q are as defined in (3.44). With this solution, the integration over in (3.32) will look like

$$I_{r_1} = \int_0^{\infty} dr_1\ r_1 \exp\left[-\frac{k\alpha_\ell r_1^2}{1+2j\alpha_\ell\eta} + \frac{jkr_1^2}{2(L-\eta)}\right] J_0\left(r_1\sqrt{p^2+q^2}\right) \quad (3.45)$$

This time making an association between (3.45) and (3.41), we get ν, a and β as

$$\nu = 0,\ a = \frac{k\alpha_\ell}{1+2j\alpha_\ell\eta} - \frac{jk}{2(L-\eta)} = k\frac{2\alpha_\ell(L-\eta) - j(1+2j\alpha_\ell\eta)}{2(1+2j\alpha_\ell\eta)(L-\eta)} = -jk\frac{1+2j\alpha_\ell L}{2(1+2j\alpha_\ell\eta)(L-\eta)}\ ,\ \beta^2 = p^2+q^2 \quad (3.46)$$

Now collecting amplitude factor terms, denoted by AF

$$AF = \frac{k^2}{2\pi(L-\eta)} 2\pi \frac{A_\ell}{1+2j\alpha_\ell\eta} j\frac{(1+2j\alpha_\ell\eta)(L-\eta)}{k(1+2j\alpha_\ell L)} = jk\frac{A_\ell}{1+2j\alpha_\ell L} \quad \text{(including the amplitude factor belonging to the summation over } \ell\text{)} \quad (3.47)$$

p^2+q^2 will be

$$p^2+q^2=\kappa^2+\frac{k^2r^2}{(L-\eta)^2}-\frac{D_{s\ell}^2}{(1+2j\alpha_\ell\eta)^2}-\frac{2k\kappa r}{L-\eta}\cos(\phi_\kappa-\phi_r)-j\frac{2\kappa D_{s\ell}}{1+2j\alpha_\ell\eta}(\cos\phi_\kappa+\sin\phi_\kappa)+j\frac{2krD_{s\ell}}{(L-\eta)(1+2j\alpha_\ell\eta)}(\cos\phi_r+\sin\phi_r) \quad (3.48)$$

$$\exp\left[-\frac{p^2+q^2}{4a}\right]=\exp\left[-j\frac{\kappa^2(1+2j\alpha_\ell\eta)(L-\eta)}{2k(1+2j\alpha_\ell L)}-j\frac{kr^2(1+2j\alpha_\ell\eta)}{2(1+2j\alpha_\ell L)(L-\eta)}+j\frac{D_{s\ell}^2(L-\eta)}{k(1+2j\alpha_\ell\eta)(1+2j\alpha_\ell L)}+j\frac{\kappa r(1+2j\alpha_\ell\eta)}{1+2j\alpha_\ell L}\cos(\phi_\kappa-\phi_r)\right.$$
$$\left.-\frac{D_{s\ell}(L-\eta)}{1+2j\alpha_\ell L}(\cos\phi_\kappa+\sin\phi_\kappa)+\frac{rD_{s\ell}}{1+2j\alpha_\ell L}(\cos\phi_r+\sin\phi_r)\right] \quad (3.49)$$

Combine $D_{s\ell}^2$ exp term of (3.38) with the third term of (3.49) to get

$$\exp\left[j\frac{D_{s\ell}^2\eta}{k(1+2j\alpha_\ell\eta)}\right]\exp\left[j\frac{D_{s\ell}^2(L-\eta)}{k(1+2j\alpha_\ell\eta)(1+2j\alpha_\ell L)}\right]=\exp\left[j\frac{D_{s\ell}^2L}{k(1+2j\alpha_\ell L)}\right] \quad (3.50)$$

Further combine $\exp\left[\frac{jkr^2}{2(L-\eta)}\right]$ outside the integration in (3.42) with second term of (3.49) to get

$$\exp\left[\frac{jkr^2}{2(L-\eta)}\right]\exp\left[-j\frac{kr^2(1+2j\alpha_\ell\eta)}{2(1+2j\alpha_\ell L)(L-\eta)}\right]=\exp\left[\left(\frac{jkr^2}{2(L-\eta)}\right)\left(1-\frac{1+2j\alpha_\ell\eta}{1+2j\alpha_\ell L}\right)\right]=\exp\left(-\frac{k\alpha_\ell r^2}{1+2j\alpha_\ell L}\right) \quad (3.51)$$

Now collect all terms to write for

$$H\left(r,\phi_r,\kappa,\phi_\kappa,\eta\right)=jk\sum_{\ell=1}^{2}\frac{A_\ell}{1+2j\alpha_\ell L}\exp\left[-\frac{k\alpha_\ell r^2}{1+2j\alpha_\ell L}-0.5j\frac{\kappa^2\left(1+2j\alpha_\ell\eta\right)\left(L-\eta\right)}{k\left(1+2j\alpha_\ell L\right)}+\frac{j\kappa r\left(1+2j\alpha_\ell\eta\right)}{1+2j\alpha_\ell L}\cos\left(\phi_\kappa-\phi_r\right)-\frac{\kappa D_{s\ell}\left(L-\eta\right)}{k\left(1+2j\alpha_\ell L\right)}\left(\cos\phi_\kappa+\sin\phi_\kappa\right)\right]$$

$$\times\exp\left[\frac{rD_{s\ell}}{1+2j\alpha_\ell L}\left(\cos\phi_r+\sin\phi_r\right)+j\frac{D_{s\ell}^2 L}{k\left(1+2j\alpha_\ell L\right)}\right]\Bigg/\sum_{\ell=1}^{2}\frac{A_\ell}{1+2j\alpha_\ell L}\exp\left[-\frac{k\alpha_\ell r^2-\left(\cos\phi_r+\sin\phi_r\right)rD_{s\ell}-jk^{-1}D_{s\ell}^2 L}{1+2j\alpha_\ell L}\right] \quad (3.52)$$

which is the same as (3.39). Now for scintillation index $m^2(\ )$, we have to compute $H(\ )H^*(\ )$ and $H(..,\kappa,..)H(..,-\kappa,..)$. Since $m^2(\ )$ will entail integration over ϕ_κ, it is best to handle this part of the integration during this process. For this, we consider only the relevant terms which are third and fourth terms of the first exponential.

A) For $H(\ )H^*(\ )$

- $$\sum_{\ell_1=1}^{2}\frac{A_{\ell_1}}{1+2j\alpha_{\ell_1}L}\sum_{\ell_2=1}^{2}\frac{A_{\ell_2}^*}{1-2j\alpha_{\ell_2}^*L}=\sum_{\ell_1=1}^{2}\sum_{\ell_2=1}^{2}\frac{A_{\ell_1}}{\left(1+2j\alpha_{\ell_1}L\right)}\frac{A_{\ell_2}^*}{\left(1-2j\alpha_{\ell_1}^*L\right)}=\sum_{\ell_1=1}^{2}\sum_{\ell_2=1}^{2}\frac{A_{\ell_1}A_{\ell_2}^*}{1+2j\left(\alpha_{\ell_1}-\alpha_{\ell_2}^*\right)L+4\alpha_{\ell_1}\alpha_{\ell_2}^*L^2} \quad (3.53)$$

- $$\exp\left(-\frac{k\alpha_{\ell_1}r^2}{1+2j\alpha_{\ell_1}L}\right)\exp\left(-\frac{k\alpha_{\ell_2}^*r^2}{1-2j\alpha_{\ell_2}^*L}\right)=\exp\left[-kr^2\frac{\alpha_{\ell_1}+\alpha_{\ell_2}^*}{1+2j\left(\alpha_{\ell_1}-\alpha_{\ell_2}^*\right)L+4\alpha_{\ell_1}\alpha_{\ell_2}^*L^2}\right] \quad (3.54)$$

- $$\exp\left[-0.5j\frac{\kappa^2\left(1+2j\alpha_{\ell_1}\eta\right)\left(L-\eta\right)}{k\left(1+2j\alpha_{\ell_1}L\right)}\right]\exp\left[0.5j\frac{\kappa^2\left(1-2j\alpha_{\ell_2}^*\eta\right)\left(L-\eta\right)}{k\left(1-2j\alpha_{\ell_2}^*L\right)}\right]=\exp\left[-\frac{\kappa^2\left(L-\eta\right)^2}{k}\frac{\alpha_{\ell_1}+\alpha_{\ell_2}^*}{1+2j\left(\alpha_{\ell_1}-\alpha_{\ell_2}^*\right)L+4\alpha_{\ell_1}\alpha_{\ell_2}^*L^2}\right] \quad (3.55)$$

- $$\exp\left[\frac{j\kappa r\left(1+2j\alpha_{\ell_1}\eta\right)}{1+2j\alpha_{\ell_1}L}\cos\left(\phi_\kappa-\phi_r\right)\right]\exp\left[-\frac{j\kappa r\left(1-2j\alpha_{\ell_2}^*\eta\right)}{1-2j\alpha_{\ell_2}^*L}\cos\left(\phi_\kappa-\phi_r\right)\right]=\exp\left[\frac{2\kappa r\left(L-\eta\right)\left(\alpha_{\ell_1}+\alpha_{\ell_2}^*\right)}{1+2j\left(\alpha_{\ell_1}-\alpha_{\ell_2}^*\right)L+4\alpha_{\ell_1}\alpha_{\ell_2}^*L^2}\cos\left(\phi_\kappa-\phi_r\right)\right] \quad (3.56)$$

- $$\exp\left[-\frac{\kappa D_{s\ell_1}(L-\eta)}{k\left(1+2j\alpha_{\ell_1}L\right)}(\cos\phi_\kappa+\sin\phi_\kappa)\right]\exp\left[-\frac{\kappa D^*_{s\ell_2}(L-\eta)}{k\left(1-2j\alpha^*_{\ell_2}L\right)}(\cos\phi_\kappa+\sin\phi_\kappa)\right]$$
$$=\exp\left\{-(\cos\phi_\kappa+\sin\phi_\kappa)\frac{\kappa(L-\eta)}{k}\left[\frac{D_{s\ell_1}\left(1-2j\alpha^*_{\ell_2}L\right)+D^*_{s\ell_2}\left(1+2j\alpha_{\ell_1}L\right)}{1+2j\left(\alpha_{\ell_1}-\alpha^*_{\ell_2}\right)L+4\alpha_{\ell_1}\alpha^*_{\ell_2}L^2}\right]\right\} \quad (3.57)$$

- $$\exp\left[\frac{rD_{s\ell_1}}{1+2j\alpha_{\ell_1}L}(\cos\phi_r+\sin\phi_r)\right]\exp\left[\frac{rD^*_{s\ell_2}}{1-2j\alpha^*_{\ell_2}L}(\cos\phi_r+\sin\phi_r)\right]=\exp\left\{r(\cos\phi_r+\sin\phi_r)\left[\frac{D_{s\ell_1}\left(1-2j\alpha^*_{\ell_2}L\right)+D^*_{s\ell_2}\left(1+2j\alpha_{\ell_1}L\right)}{1+2j\left(\alpha_{\ell_1}-\alpha^*_{\ell_2}\right)L+4\alpha_{\ell_1}\alpha^*_{\ell_2}L^2}\right]\right\} \quad (3.58)$$

- $$\exp\left[j\frac{D^2_{s\ell}L}{k\left(1+2j\alpha_{\ell_1}L\right)}\right]\exp\left[-j\frac{\left(D^2_{s\ell_2}\right)^*L}{k\left(1-2j\alpha^*_{\ell_2}L\right)}\right]=\exp\left\{jL\frac{D^2_{s\ell_1}\left(1-2j\alpha^*_{\ell_2}L\right)-\left(D^2_{s\ell_2}\right)^*\left(1+2j\alpha_{\ell_1}L\right)}{k\left[1+2j\left(\alpha_{\ell_1}-\alpha^*_{\ell_2}\right)L+4\alpha_{\ell_1}\alpha^*_{\ell_2}L^2\right]}\right\} \quad (3.59)$$

Now arrange for integration over ϕ_κ such that

$$\int_0^{2\pi} d\phi_\kappa \exp\left[\kappa\left(p_1\cos\phi_\kappa+q_1\sin\phi_\kappa\right)\right]=2\pi I_0\left(\kappa\sqrt{p_1^2+q_1^2}\right) \quad (3.60)$$

By associating (3.56) and (3.57) with (3.60), we identify p_1 and q_1 as

$$p_1=\frac{2r(L-\eta)\left(\alpha_{\ell_1}+\alpha^*_{\ell_2}\right)}{1+2j\left(\alpha_{\ell_1}-\alpha^*_{\ell_2}\right)L+4\alpha_{\ell_1}\alpha^*_{\ell_2}L^2}\cos\phi_r-\frac{L-\eta}{k}\left[\frac{D_{s\ell_1}\left(1-2j\alpha^*_{\ell_2}L\right)+D^*_{s\ell_2}\left(1+2j\alpha_{\ell_1}L\right)}{1+2j\left(\alpha_{\ell_1}-\alpha^*_{\ell_2}\right)L+4\alpha_{\ell_1}\alpha^*_{\ell_2}L^2}\right]$$
$$q_1=\frac{2r(L-\eta)\left(\alpha_{\ell_1}+\alpha^*_{\ell_2}\right)}{1+2j\left(\alpha_{\ell_1}-\alpha^*_{\ell_2}\right)L+4\alpha_{\ell_1}\alpha^*_{\ell_2}L^2}\sin\phi_r-\frac{L-\eta}{k}\left[\frac{D_{s\ell_1}\left(1-2j\alpha^*_{\ell_2}L\right)+D^*_{s\ell_2}\left(1+2j\alpha_{\ell_1}L\right)}{1+2j\left(\alpha_{\ell_1}-\alpha^*_{\ell_2}\right)L+4\alpha_{\ell_1}\alpha^*_{\ell_2}L^2}\right] \quad (3.61)$$

This way $\left(p_1^2+q_1^2\right)^{1/2}$ will be

$$\left(p_1^2+q_1^2\right)^{1/2}=\frac{(L-\eta)}{1+2j\left(\alpha_{\ell_1}-\alpha_{\ell_2}^*\right)L+4\alpha_{\ell_1}\alpha_{\ell_2}^*L^2}\left\{4r^2\left(\alpha_{\ell_1}+\alpha_{\ell_2}^*\right)^2+\frac{2}{k^2}\left[D_{s\ell_1}^2\left(1-2j\alpha_{\ell_2}^*L\right)^2+\left(D_{s\ell_2}^2\right)^*\left(1+2j\alpha_{\ell_1}L\right)^2\right]\right.$$

$$\left.+\frac{4D_{s\ell_1}D_{s\ell_2}^*}{k^2}\left[1+2j\left(\alpha_{\ell_1}-\alpha_{\ell_2}^*\right)L+4\alpha_{\ell_1}\alpha_{\ell_2}^*L^2\right]-\frac{4r}{k}\left(\alpha_{\ell_1}+\alpha_{\ell_2}^*\right)(\cos\phi_r+\sin\phi_r)\left[D_{s\ell_1}\left(1-2j\alpha_{\ell_2}^*L\right)+D_{s\ell_2}^*\left(1+2j\alpha_{\ell_1}L\right)\right]\right\}^{1/2} \quad (3.62)$$

Now we continue with terms of $H(..,\kappa,..)H(..,-\kappa,..)$

B) For $H(..,\kappa,..)H(..,-\kappa,..)$

- $$\sum_{\ell_1=1}^{2}\frac{A_{\ell_1}}{1+2j\alpha_{\ell_1}L}\sum_{\ell_1=1}^{2}\frac{A_{\ell_2}}{1+2j\alpha_{\ell_2}L}=\sum_{\ell_1=1}^{2}\sum_{\ell_2=1}^{2}\frac{A_{\ell_1}A_{\ell_2}}{1+2j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)L-4\alpha_{\ell_1}\alpha_{\ell_2}L^2} \quad (3.63)$$

- $$\exp\left(-\frac{k\alpha_{\ell_1}r^2}{1+2j\alpha_{\ell_1}L}\right)\exp\left(-\frac{k\alpha_{\ell_2}r^2}{1+2j\alpha_{\ell_2}L}\right)=\exp\left[-kr^2\frac{\alpha_{\ell_1}+\alpha_{\ell_2}+4j\alpha_{\ell_1}\alpha_{\ell_2}L}{1+2j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)L-4\alpha_{\ell_1}\alpha_{\ell_2}L^2}\right] \quad (3.64)$$

- $$\exp\left[-0.5j\frac{\kappa^2\left(1+2j\alpha_{\ell_1}\eta\right)(L-\eta)}{k\left(1+2j\alpha_{\ell_1}L\right)}\right]\exp\left[-0.5j\frac{\kappa^2\left(1+2j\alpha_{\ell_2}\eta\right)(L-\eta)}{k\left(1+2j\alpha_{\ell_2}L\right)}\right]=\exp\left\{-j\frac{\kappa^2(L-\eta)}{k}\left[\frac{1+j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)(L+\eta)-4\alpha_{\ell_1}\alpha_{\ell_2}\eta L}{1+2j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)L-4\alpha_{\ell_1}\alpha_{\ell_2}L^2}\right]\right\} \quad (3.65)$$

- $$\exp\left[\frac{j\kappa r\left(1+2j\alpha_{\ell_1}\eta\right)}{1+2j\alpha_{\ell_1}L}\cos(\phi_\kappa-\phi_r)\right]\exp\left[-\frac{j\kappa r\left(1+2j\alpha_{\ell_2}\eta\right)}{1+2j\alpha_{\ell_2}L}\cos(\phi_\kappa-\phi_r)\right]=\exp\left[\frac{2\kappa r(L-\eta)\left(\alpha_{\ell_1}-\alpha_{\ell_2}\right)}{1+2j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)L-4\alpha_{\ell_1}\alpha_{\ell_2}L^2}\cos(\phi_\kappa-\phi_r)\right] \quad (3.66)$$

- $$\exp\left[-\frac{\kappa D_{s\ell_1}(L-\eta)}{k\left(1+2j\alpha_{\ell_1}L\right)}(\cos\phi_\kappa+\sin\phi_\kappa)\right]\exp\left[-\frac{\kappa D_{s\ell_2}(L-\eta)}{k\left(1+2j\alpha_{\ell_2}L\right)}(\cos\phi_\kappa+\sin\phi_\kappa)\right]$$
$$=\exp\left\{-(\cos\phi_\kappa+\sin\phi_\kappa)\frac{\kappa(L-\eta)}{k}\left[\frac{D_{s\ell_1}\left(1+2j\alpha_{\ell_2}L\right)-D_{s\ell_2}\left(1+2j\alpha_{\ell_1}L\right)}{1+2j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)L-4\alpha_{\ell_1}\alpha_{\ell_2}L^2}\right]\right\} \quad (3.67)$$

- $$\exp\left[\frac{rD_{s\ell_1}}{1+2j\alpha_{\ell_1}L}(\cos\phi_r+\sin\phi_r)\right]\exp\left[\frac{rD_{s\ell_2}}{1+2j\alpha_{\ell_2}L}(\cos\phi_r+\sin\phi_r)\right]=\exp\left\{r(\cos\phi_r+\sin\phi_r)\left[\frac{D_{s\ell_1}\left(1+2j\alpha_{\ell_2}L\right)+D_{s\ell_2}\left(1+2j\alpha_{\ell_1}L\right)}{1+2j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)L-4\alpha_{\ell_1}\alpha_{\ell_2}L^2}\right]\right\} \quad (3.68)$$

- $$\exp\left[j\frac{D_{s\ell}^2L}{k\left(1+2j\alpha_{\ell_1}L\right)}\right]\exp\left[j\frac{D_{s\ell_2}^2L}{k\left(1+2j\alpha_{\ell_2}L\right)}\right]=\exp\left\{jL\frac{D_{s\ell_1}^2\left(1+2j\alpha_{\ell_2}L\right)+D_{s\ell_2}^2\left(1+2j\alpha_{\ell_1}L\right)}{k\left[1+2j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)L-4\alpha_{\ell_1}\alpha_{\ell_2}L^2\right]}\right\} \quad (3.69)$$

Now arrange for integration over ϕ_κ such that

$$\int_0^{2\pi} d\phi_\kappa \exp\left[\kappa\left(p_2\cos\phi_\kappa+q_2\sin\phi_\kappa\right)\right]=2\pi I_0\left(\kappa\sqrt{p_2^2+q_2^2}\right) \quad (3.70)$$

By associating (3.66) and (3.67) with (3.70), we identify p_2 and q_2 as

$$p_2=\frac{2r(L-\eta)\left(\alpha_{\ell_1}-\alpha_{\ell_2}\right)}{1+2j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)L-4\alpha_{\ell_1}\alpha_{\ell_2}L^2}\cos\phi_r-\frac{L-\eta}{k}\left[\frac{D_{s\ell_1}\left(1+2j\alpha_{\ell_2}L\right)-D_{s\ell_2}\left(1+2j\alpha_{\ell_1}L\right)}{1+2j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)L-4\alpha_{\ell_1}\alpha_{\ell_2}L^2}\right]$$
$$q_2=\frac{2r(L-\eta)\left(\alpha_{\ell_1}-\alpha_{\ell_2}\right)}{1+2j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)L-4\alpha_{\ell_1}\alpha_{\ell_2}L^2}\sin\phi_r-\frac{L-\eta}{k}\left[\frac{D_{s\ell_1}\left(1+2j\alpha_{\ell_2}L\right)-D_{s\ell_2}\left(1+2j\alpha_{\ell_1}L\right)}{1+2j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)L-4\alpha_{\ell_1}\alpha_{\ell_2}L^2}\right] \quad (3.71)$$

This way $\left(p_2^2+q_2^2\right)^{1/2}$ will be

$$\left(p_2^2+q_2^2\right)^{1/2}=\frac{(L-\eta)}{1+2j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)L-4\alpha_{\ell_1}\alpha_{\ell_2}L^2}\left\{4r^2\left(\alpha_{\ell_1}-\alpha_{\ell_2}\right)^2+\frac{2}{k^2}\left[D_{s\ell_1}^2\left(1+2j\alpha_{\ell_2}L\right)^2+D_{s\ell_2}^2\left(1+2j\alpha_{\ell_1}L\right)^2\right]\right.$$
$$\left.-\frac{4D_{s\ell_1}D_{s\ell_2}}{k^2}\left[1+2j\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)L-4\alpha_{\ell_1}\alpha_{\ell_2}L^2\right]-\frac{4r}{k}\left(\alpha_{\ell_1}-\alpha_{\ell_2}\right)\left(\cos\phi_r+\sin\phi_r\right)\left[D_{s\ell_1}\left(1+2j\alpha_{\ell_2}L\right)-D_{s\ell_2}\left(1+2j\alpha_{\ell_1}L\right)\right]\right\}^{1/2} \quad (3.72)$$

By using

$$m^2(\mathbf{r},L)=m^2(r,\phi_r,L)=2\left\langle\psi_1(\mathbf{r},L)\psi_1^*(\mathbf{r},L)\right\rangle+2\,\mathrm{Re}\left[\left\langle\psi_1(\mathbf{r},L)\psi_1(\mathbf{r},L)\right\rangle\right]$$
$$=4\pi\int_0^L d\eta\int_0^{2\pi}d\phi_\kappa\int_0^\infty d\kappa\kappa\,\Phi_n(\kappa)\ \left\{\left|H(r,\phi_r,\kappa,\phi_\kappa,\eta)\right|^2+\mathrm{Re}\left[H(r,\phi_r,\kappa,\phi_\kappa,\eta)H(r,\phi_r,-\kappa,\phi_\kappa,\eta)\right]\right\} \quad (3.73)$$

After inserting for $H(\ )$ in (3.73) from (3.52), using von-Karman spectrum and then performing the integration over ϕ_κ with the help of (3.70), then scintillation index expression in (3.73) for sinusoidal hyperbolic Gaussian beam will become

$$
\begin{aligned}
m^2(r,\phi_r,L) = {} & 2.6056k^2C_n^2\Bigg\{\sum_{\ell_1=1}^{N}\sum_{\ell_1=1}^{N}\frac{A_{\ell_1}A_{\ell_2}^*}{C_{da}}\exp\left(-kr^2\frac{\alpha_{\ell_1}+\alpha_{\ell_2}^*}{C_{da}}\right)\exp\left[r\left(\sin\phi_r+\cos\phi_r\right)\frac{D_{s\ell_1}\left(1-2j\alpha_{\ell_2}^*L\right)+D_{s\ell_2}^*\left(1+2j\alpha_{\ell_1}L\right)}{C_{da}}\right] \\
& \times\exp\left[jL\frac{D_{s\ell_1}^2\left(1-2j\alpha_{\ell_2}^*L\right)-\left(D_{s\ell_2}^*\right)^2\left(1+2j\alpha_{\ell_1}L\right)}{kC_{da}}\right]\int_0^L d\eta\int_0^\infty d\kappa\kappa\frac{\exp\left[-\ell_0^2\kappa^2/35.05\right]}{\left(\kappa^2+4\pi^2/L_0^2\right)^{11/6}}\exp\left[-\frac{\kappa^2\left(L-\eta\right)^2\left(\alpha_{\ell_1}+\alpha_{\ell_2}^*\right)}{kC_{da}}\right] \\
& I_0\Bigg(\kappa\frac{L-\eta}{C_{da}}\Big\{4r^2\left(\alpha_{\ell_1}+\alpha_{\ell_2}^*\right)^2+4\frac{D_{s\ell_1}D_{s\ell_2}^*C_{da}}{k^2}+\frac{2}{k^2}\left[D_{s\ell_1}^2\left(1-2j\alpha_{\ell_2}^*L\right)+\left(D_{s\ell_2}^*\right)^2\left(1+2j\alpha_{\ell_1}L\right)\right] \\
& -\frac{4r_o}{k}\left(\alpha_{\ell_1}+\alpha_{\ell_2}^*\right)\left(\sin\phi_r+\cos\phi_r\right)\left[D_{s\ell_1}\left(1-2j\alpha_{\ell_2}^*L\right)-D_{s\ell_2}^*\left(1+2j\alpha_{\ell_1}L\right)\right]\Big\}^{1/2}\Bigg)/\left|U_0\left(r,\phi_r,L\right)\right|^2 \\
& -\mathrm{Re}\Bigg[\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\frac{A_{\ell_1}A_{\ell_2}}{C_{dc}}\exp\left(-kr^2\frac{\alpha_{\ell_1}+\alpha_{\ell_2}+4j\alpha_{\ell_1}\alpha_{\ell_2}L}{C_{dc}}\right)\exp\left[jL\frac{D_{s\ell_1}^2\left(1+2j\alpha_{\ell_2}L\right)+D_{s\ell_2}^2\left(1+2j\alpha_{\ell_1}L\right)}{kC_{dc}}\right] \\
& \times\exp\left[r\left(\sin\phi_r+\cos\phi_r\right)\frac{D_{s\ell_1}\left(1+2j\alpha_{\ell_2}L\right)+D_{s\ell_2}\left(1+2j\alpha_{\ell_1}L\right)}{C_{dc}}\right]\int_0^L d\eta\int_0^\infty d\kappa\kappa\frac{\exp\left[-\ell_0^2\kappa^2/35.05\right]}{\left(\kappa^2+4\pi^2/L_0^2\right)^{11/6}} \\
& \times\exp\left\{-j\frac{\kappa^2\left(L-\eta\right)}{k}\frac{\left[1+j\left(L+\eta\right)\left(\alpha_{\ell_1}+\alpha_{\ell_2}\right)-4\alpha_{\ell_1}\alpha_{\ell_2}\eta L\right]}{C_{dc}}\right\}I_0\Bigg(\kappa\frac{L-\eta}{C_{dc}}\Big\{4r^2\left(\alpha_{\ell_1}-\alpha_{\ell_2}\right)^2+2\frac{\left[D_{s\ell_1}-D_{s\ell_2}+2j\left(D_{s_{\ell_1}}\alpha_{\ell_2}-D_{s\ell_2}\alpha_{\ell_1}\right)L\right]^2}{k^2} \\
& -4r\left(\alpha_{\ell_1}-\alpha_{\ell_2}\right)\left(\sin\phi_r+\cos\phi_r\right)\frac{\left[D_{s\ell_1}-D_{s\ell_2}+2j\left(D_{s\ell_1}\alpha_{\ell_2}-D_{s\ell_2}\alpha_{\ell_1}\right)L\right]}{k}\Big\}^{1/2}\Bigg)/U_0^2\left(r,\phi_r,L\right)\Bigg]\Bigg\} \qquad (3.74)
\end{aligned}
$$

where

$$C_{da} = 1 + 2j\left(\alpha_{\ell_1} - \alpha^*_{\ell_2}\right)L + 4\alpha_{\ell_1}\alpha^*_{\ell_2}L^2$$

$$C_{dc} = 1 + 2j\left(\alpha_{\ell_1} + \alpha_{\ell_2}\right)L - 4\alpha_{\ell_1}\alpha_{\ell_2}L^2 \quad (3.75)$$

ℓ_0 : Inner scale of turbulence, L_0: Outer scale of turbulence

Lab exercise, the above is given in Scin_SinoHyp_L.m MATLAB file on the course webpage. Find and compare scintillation values from this m file and to the graphs provided in A15 and A16.

4. Scintillation Formulation via Extended Huygens-Fresnel Integral

For this method we go back to (R23) which is

$$m^2(\mathbf{r},L) = \frac{\left\langle I^2(\mathbf{r},L)\right\rangle}{\left\langle I(\mathbf{r},L)\right\rangle^2} - 1 \qquad (4.1)$$

which means that we have to evaluate $\left\langle I^2(\mathbf{r},L)\right\rangle$, i.e. average squared intensity (on receiver plane) and $\left\langle I(\mathbf{r},L)\right\rangle$, i.e. average intensity, The latter, $\left\langle I(\mathbf{r},L)\right\rangle$ is relatively easy and given by (1.6) or (1.7), while $\left\langle I^2(\mathbf{r},L)\right\rangle$ is relatively difficult, since in this case we need $\psi(\mathbf{r},\mathbf{s})$ in the fourth order, i.e., $\left\langle \psi(\mathbf{r},\mathbf{s_1}) + \psi^*(\mathbf{r},\mathbf{s_2}) + \psi(\mathbf{r},\mathbf{s_3}) + \psi^*(\mathbf{r},\mathbf{s_4})\right\rangle$. Note that in $\left\langle I(\mathbf{r},L)\right\rangle$, second order of $\psi(\mathbf{r},\mathbf{s})$, i.e. $\left\langle \psi(\mathbf{r},\mathbf{s_1}) + \psi^*(\mathbf{r},\mathbf{s_2})\right\rangle$ is used as also apparent from (1.6) and (1.7). From the literature, we get the fourth order moment (in Cartesian coordinates) as (only the x part is shown)

$$\begin{aligned}
&\left\langle \exp\left[\psi(\mathbf{r},\mathbf{s_1}) + \psi^*(\mathbf{r},\mathbf{s_2}) + \psi(\mathbf{r},\mathbf{s_3}) + \psi^*(\mathbf{r},\mathbf{s_4})\right]\right\rangle = \exp\left[-\frac{j}{\rho_{\chi s}^2}\left(s_{1x}^2 - s_{2x}^2 + s_{3x}^2 - s_{4x}^2 - 2s_{1x}s_{3x} + 2s_{2x}s_{4x}\right)\right] \\
&\quad \times \left\{ \exp\left[-\frac{1}{\rho_t^2}\left(s_{1x}^2 + s_{2x}^2 + s_{3x}^2 + s_{4x}^2 - 2s_{1x}s_{2x} + 2s_{1x}s_{3x} - 2s_{1x}s_{4x} - 2s_{2x}s_{3x} + 2s_{2x}s_{4x} - 2s_{3x}s_{4x}\right)\right] \right. \\
&\quad + 2\sigma_\chi^2 \exp\left[-\frac{1}{\rho_t^2}\left(2s_{1x}^2 + s_{2x}^2 + 2s_{3x}^2 + s_{4x}^2 - 2s_{1x}s_{2x} - 2s_{1x}s_{4x} - 2s_{2x}s_{3x} + 2s_{2x}s_{4x} - 2s_{3x}s_{4x}\right)\right] \\
&\quad \left. + 2\sigma_\chi^2 \exp\left[-\frac{1}{\rho_t^2}\left(s_{1x}^2 + 2s_{2x}^2 + s_{3x}^2 + 2s_{4x}^2 - 2s_{1x}s_{2x} + 2s_{1x}s_{3x} - 2s_{1x}s_{4x} - 2s_{2x}s_{3x} - 2s_{3x}s_{4x}\right)\right]\right\} \qquad (4.2)
\end{aligned}$$

In the classic approach, we would attempt to derive $\left\langle I^2(\mathbf{r},L)\right\rangle$ and $\left\langle I(\mathbf{r},L)\right\rangle$ by hand and thus obtain analytic expressions. While this is valuable, it is also difficult, particularly in the case of hand derivation of $\left\langle I^2(\mathbf{r},L)\right\rangle$. We could use numeric integration, but then the correct estimation of lower and upper limits (to be used instead of minus infinity and plus infinity) create problems, besides $\left\langle I^2(\mathbf{r},L)\right\rangle$ entails at least quadruple integration.

To solve these problems, we have developed a semi-analytic method, which we now explain.

Let's take either the x (one dimensional) part of $\left\langle I^2(\mathbf{r},L)\right\rangle$ or the whole of $\left\langle I(\mathbf{r},L)\right\rangle$, for the beams within our interest, the quadruple integral in these two cases would look like

$$I=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_1dt_2dt_3dt_4\exp\left(-q_{11}t_1^2-q_{22}t_2^2-q_{33}t_3^2-q_{44}t_4^2\right)\exp\left(-2q_{12}t_1t_2-2q_{13}t_1t_3-2q_{14}t_1t_4-2q_{23}t_2t_3-2q_{24}t_2t_4-2q_{34}t_3t_4\right)$$
$$\times\exp\left(-2r_1t_1-2r_2t_2-2r_3t_3-2r_4t_4\right)t_1^{n_1}t_2^{n_2}t_3^{n_3}t_4^{n_4} \qquad (4.3)$$

To reduce the quadruple integral of (4.3) to the case of a single integral, initially we isolate one integral, namely the one with respect to t_1 as follows

$$I=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_2dt_3dt_4\exp\left(-q_{22}t_2^2-q_{33}t_3^2-q_{44}t_4^2-2q_{23}t_2t_3-2q_{24}t_2t_4-2q_{34}t_3t_4\right)\exp\left(-2r_2t_2-2r_3t_3-2r_4t_4\right)t_2^{n_2}t_3^{n_3}t_4^{n_4}\int_{-\infty}^{\infty} dt_1\exp\left(-q_{11}t_1^2-2r_gt_1\right)t_1^{n_1} \qquad (4.4)$$

where $r_g = r_1 + q_{12}t_2 + q_{13}t_3 + q_{14}t_4$. Now using a modified version 3.462.2 of [I. S. Gradshteyn, I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, 2000)] which is

$$\int_{-\infty}^{\infty} dt\, \exp\left(-qt^2 - 2rt\right)t^n = n!\exp\left(\frac{r^2}{q}\right)\sum_{i=0}^{\lfloor n/2\rfloor}\frac{\Gamma\left(i+1/2\right)}{q^{i+1/2}(n-2i)!(2i)!}\left(-\frac{r}{q}\right)^{n-2i} \qquad (4.5)$$

The isolated integral in (4.4) can be solved, hence (4.4) will reduce to the following triple integral

$$\begin{aligned} I = \left(-1\right)^{n_1} n_1!\exp\left(\frac{r_1^2}{q_{11}}\right)\sum_{i=0}^{\lfloor n_1/2\rfloor}\frac{\Gamma\left(i+1/2\right)}{q_{11}^{n_1-i+1/2}(n_1-2i)!(2i)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_2 dt_3 dt_4 \exp\left[-\left(q_{22}-\frac{q_{12}^2}{q_{11}}\right)t_2^2 - \left(q_{33}-\frac{q_{13}^2}{q_{11}}\right)t_3^2 - \left(q_{44}-\frac{q_{14}^2}{q_{11}}\right)t_4^2\right] \\ \times\exp\left[-2\left(q_{23}-\frac{q_{12}q_{13}}{q_{11}}\right)t_2t_3 - 2\left(q_{24}-\frac{q_{12}q_{14}}{q_{11}}\right)t_2t_4 - 2\left(q_{34}-\frac{q_{13}q_{14}}{q_{11}}\right)t_3t_4\right]\exp\left[-2\left(r_2-\frac{r_1q_{12}}{q_{11}}\right)t_2 - 2\left(r_3-\frac{r_1q_{13}}{q_{11}}\right)t_3 - 2\left(r_4-\frac{r_1q_{14}}{q_{11}}\right)t_4\right] \\ \times t_2^{n_2}t_3^{n_3}t_4^{n_4}\left(r_1+q_{12}t_2+q_{13}t_3+q_{14}t_4\right)^{n_1-2i} \end{aligned} \qquad (4.6)$$

The last term of (4.6) can be expanded via Binomial formula, and the result can be rearranged so that integral with respect to t_2 can be managed individually again by the use of (4.5). The development continues in this manner until all the integrations have been performed. To facilitate an easy track of equation development, the Matlab function ExpPolyHerm4 is organized as the main function plus the others named ExpPoly4, ExpPoly3, ExpPoly2, ExpPoly1, which call each other in numeric sequence to transform the quadruple integral into triple, quadruple and single integrals, while the main function ExpPolyHerm4 initiates the first call and makes the final evaluation. The Hermite polynomials can be handled by writing for their series expansions and embedding the arising powers of t_1, t_2, t_3 and t_4 into n_1, n_2, n_3 and n_4 in (4.3).

Below we illustrate the use of this semi-analytic method. For this, we take the Cartesian coordinate representation of (lowest order) Sinusoidal Hyperbolic Gaussian beam which is

$$u_s\left(s_x, s_y\right) = \sum_{\ell=1}^{N} A_\ell \exp\left[-\left(0.5k\alpha_{x\ell} s_x^2 - D_{x\ell} s_x\right)\right] \exp\left[-\left(0.5k\alpha_{y\ell} s_y^2 - D_{y\ell} s_y\right)\right] \quad (4.7)$$

After adding partial coherence to this beam, we get the mutual coherence function of the source as

$$\Gamma_s\left(s_{1x}, s_{1y}, s_{2x}, s_{2y}\right) = \exp\left(-\frac{s_{1x}^2 + s_{2x}^2 + s_{1y}^2 + s_{2y}^2 - 2s_{1x}s_{2x} - 2s_{1y}s_{2y}}{2\sigma_s^2}\right) \sum_{\ell_1=1}^{N} A_{\ell_1} \exp\left[-\left(0.5k\alpha_{x\ell_1} s_{1x}^2 - D_{x\ell_1} s_{1x}\right)\right] \exp\left[-\left(0.5k\alpha_{y\ell_1} s_{1y}^2 - D_{y\ell_1} s_{1y}\right)\right]$$
$$\times \sum_{\ell_2=1}^{N} A_{\ell_2}^* \exp\left[-\left(0.5k\alpha_{x\ell_2}^* s_{2x}^2 - D_{x\ell_2}^* s_{2x}\right)\right] \exp\left[-\left(0.5k\alpha_{y\ell_2}^* s_{2y}^2 - D_{y\ell_2}^* s_{2y}\right)\right] \quad (4.8)$$

The corresponding extended Huygens-Fresnel integral is

$$\left\langle I\left(r_x, r_y, L\right)\right\rangle = \left(\frac{k}{2\pi L}\right)^2 \sum_{\ell_1=1}^{N} A_{\ell_1} \sum_{\ell_2=1}^{N} A_{\ell_2}^* \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ds_{1x} ds_{1y} ds_{2x} ds_{2y} \exp\left(-\frac{s_{1x}^2 + s_{2x}^2 + s_{1y}^2 + s_{2y}^2 - 2s_{1x}s_{2x} - 2s_{1y}s_{2y}}{2\sigma_s^2}\right)$$
$$\times \exp\left[-\left(0.5k\alpha_{x\ell_1} s_{1x}^2 - D_{x\ell_1} s_{1x} + 0.5k\alpha_{y\ell_1} s_{1y}^2 - D_{y\ell_1} s_{1y}\right)\right] \exp\left[-\left(0.5k\alpha_{x\ell_2}^* s_{2x}^2 - D_{x\ell_2}^* s_{2x} + 0.5k\alpha_{y\ell_2}^* s_{2y}^2 - D_{y\ell_2}^* s_{2y}\right)\right]$$
$$\times \exp\left[\frac{jk}{2L}\left(s_{1x}^2 - 2r_x s_{1x} + s_{1y}^2 - 2r_y s_{1y} - s_{2x}^2 + 2r_x s_{2x} - s_{2y}^2 + 2r_y s_{2y}\right)\right] \exp\left(-\frac{s_{1x}^2 + s_{2x}^2 + s_{1y}^2 + s_{2y}^2 - 2s_{1x}s_{2x} - 2s_{1y}s_{2y}}{\rho_t^2}\right) \quad (4.9)$$

Now all we have to do is prepare the following matrices

$$\mathbf{Q}=\begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{12} & q_{22} & q_{23} & q_{24} \\ q_{13} & q_{23} & q_{33} & q_{34} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{pmatrix} \ , \ \mathbf{R}=\begin{pmatrix} r_1 & r_2 & r_3 & r_4 \end{pmatrix} \ , \ \mathbf{N}=\begin{pmatrix} n_1 & n_2 & n_3 & n_4 \end{pmatrix} \ , \ \mathbf{M}=\begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} \qquad (4.10)$$

where the elements are to be determined by making an association between (4.3) and (4.9) and then writing the following equivalences

$$q_{11}=0.5k\alpha_{x\ell_1}+\frac{1}{2\sigma_s^2}-\frac{jk}{2L}+\frac{1}{\rho_t^2} \quad , \quad q_{12}=-\frac{1}{2\sigma_s^2}-\frac{1}{\rho_t^2}=q_{34} \quad , \quad q_{13}=q_{14}=0$$

$$q_{22}=0.5k\alpha^*_{x\ell_2}+\frac{1}{2\sigma_s^2}+\frac{jk}{2L}+\frac{1}{\rho_t^2} \quad , \quad q_{23}=q_{24}=0$$

$$q_{33}=0.5k\alpha_{y\ell_1}+\frac{1}{2\sigma_s^2}-\frac{jk}{2L}+\frac{1}{\rho_t^2} \quad , \quad q_{44}=0.5k\alpha^*_{y\ell_2}+\frac{1}{2\sigma_s^2}+\frac{jk}{2L}+\frac{1}{\rho_t^2} \qquad (4.11)$$

$$r_1=-0.5D_{x\ell_1}+\frac{jkr_x}{2L} \ , \ r_2=-0.5D^*_{x\ell_2}-\frac{jkr_x}{2L} \ , \ r_3=-0.5D_{y\ell_1}+\frac{jkr_y}{2L} \ , \ r_4=-0.5D^*_{y\ell_2}-\frac{jkr_y}{2L} \qquad (4.12)$$

$$n_1=n_2=n_3=n_4=0 \qquad m_1=m_2=m_3=m_4=0 \qquad c_1=c_2=c_3=c_4=0 \qquad (4.13)$$

Note that matrix M refers to the existence of Hermite polynomials, i.e. higher orders in the source expression of (4.7). In the present case, since we deal with lowest order, all matrix elements of M are zero.

Exercise 4.1 : Now we turn to a MATLAB exercise. On the course webpage you will find the m code Sino_Hyp_Her4.m which plots the source, that is $I_s\left(s_x,s_y\right)=\Gamma_s\left(s_x=s_{1x}=s_{2x},s_y=s_{1y}=s_{2y}\right)$ and receiver, that is $\left\langle I\left(r_x,r_y,L\right)\right\rangle$, intensity profiles along the slanted axis. Run this m file for all beam types (4.7) is able to generate, i.e. Gaussian, cos Gaussian, cosh Gaussian, sine Gaussian, sinh Gaussian, annular Gaussian beams, comparing at least two readings of each beam intensity levels with those produced by ParCoh_SinoHypR_tur.m or Transmittance635.m at the same source and propagation settings. Note that to evaluate $\left\langle I\left(r_x, r_y, L\right)\right\rangle$ by the semi-analytic method, you have to download from the course webpage the m files, exppoly1.m, exppoly2.m, exppoly3.m, exppoly4.m, ExpPolyHermLagu401.m which can handle source beams incorporating Laguerre polynomials as well.

Evaluation of $\left\langle I^2(\mathbf{r},L)\right\rangle$ via semi-analytic method

For this we initially write for $\left\langle I^2(\mathbf{r},L)\right\rangle$ for the x part only, which is

$$\left\langle I^2(r_x,L)\right\rangle = \frac{k^4}{(2\pi L)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ds_{1x}ds_{2x}ds_{3x}ds_{4x}\Gamma_s(s_{1x},s_{2x},s_{3x},s_{4x})\exp\left[\frac{jk}{2L}\left(s_{1x}^2 - s_{2x}^2 + s_{3x}^2 - s_{4x}^2 - 2r_x s_{1x} + 2r_x s_{2x} - 2r_x s_{3x} + 2r_x s_{4x}\right)\right]$$
$$\times\exp\left[-\frac{j}{\rho_{\chi s}^2}\left(s_{1x}^2 - s_{2x}^2 + s_{3x}^2 - s_{4x}^2 - 2s_{1x}s_{3x} + 2s_{2x}s_{4x}\right)\right]\left\{\exp\left[-\frac{1}{\rho_t^2}\left(s_{1x}^2 + s_{2x}^2 + s_{3x}^2 + s_{4x}^2 - 2s_{1x}s_{2x} + 2s_{1x}s_{3x} - 2s_{1x}s_{4x} - 2s_{2x}s_{3x} + 2s_{2x}s_{4x} - 2s_{3x}s_{4x}\right)\right]\right.$$
$$+2\sigma_\chi^2\exp\left[-\frac{1}{\rho_t^2}\left(2s_{1x}^2 + s_{2x}^2 + 2s_{3x}^2 + s_{4x}^2 - 2s_{1x}s_{2x} - 2s_{1x}s_{4x} - 2s_{2x}s_{3x} + 2s_{2x}s_{4x} - 2s_{3x}s_{4x}\right)\right]$$
$$\left.+2\sigma_\chi^2\exp\left[-\frac{1}{\rho_t^2}\left(s_{1x}^2 + 2s_{2x}^2 + s_{3x}^2 + 2s_{4x}^2 - 2s_{1x}s_{2x} + 2s_{1x}s_{3x} - 2s_{1x}s_{4x} - 2s_{2x}s_{3x} - 2s_{3x}s_{4x}\right)\right]\right\} \qquad (4.14)$$

where for a Kolmogorov spectrum, that is $\Phi_n(\kappa) = 0.033C_n^2\kappa^{-11/3}$, $\sigma_\chi^2 = 0.124C_n^2k^{7/6}L^{11/6}$, $\rho_{\chi S} = \left(0.1134k^{13/6}C_n^2L^{5/6}\right)^{-1/2}$. (4.14) is in the form of

$\left\langle I^2(r_x,L)\right\rangle = I_{Ax} + I_{Bx} + I_{Cx}$. This means $\left\langle I^2(r_y,L)\right\rangle$ will be

$\left\langle I^2(r_y,L)\right\rangle = I_{Ay} + I_{By} + I_{Cy}$. Then $\left\langle I^2(\mathbf{r},L)\right\rangle$ will be obtained from

$$\left\langle I^2(\mathbf{r},L)\right\rangle = \left\langle I^2(r_x,L)\right\rangle \bullet \left\langle I^2(r_y,L)\right\rangle = \left(I_{Ax} + I_{Bx} + I_{Cx}\right)\bullet\left(I_{Ay} + I_{By} + I_{Cy}\right) = I_{Ax}I_{Ay} + I_{Bx}I_{By} + I_{Cx}I_{Cy} \qquad (4.15)$$

Hence $\left\langle I^2(\mathbf{r},L)\right\rangle$ can be evaluated in two steps once by calling ExpPolyHermLagu401.m for *x* part, then calling the same for *y* part. Note also that (4.15) points to a dot product multiplication. Now by inserting the *x* part of Sinusoidal Hyperbolic Gaussian beam for $\Gamma_s(s_{1x},s_{2x},s_{3x},s_{4x})$, (4.14) will become

$$\left\langle I^2(r_x,L)\right\rangle = \frac{k^4}{(2\pi L)^4}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\sum_{\ell_3=1}^{N}\sum_{\ell_4=1}^{N} A_{\ell_1}A_{\ell_2}^{*}A_{\ell_3}A_{\ell_4}^{*}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ds_{1x}ds_{2x}ds_{3x}ds_{4x}\exp\left(-\frac{s_{1x}^2+s_{2x}^2+s_{3x}^2+s_{4x}^2-2s_{1x}s_{2x}-2s_{3x}s_{4x}}{2\sigma_s^2}\right)$$
$$\times\exp\left[-\left(0.5k\alpha_{x\ell_1}s_{1x}^2+0.5k\alpha_{x\ell_2}^{*}s_{2x}^2+0.5k\alpha_{x\ell_3}s_{3x}^2+0.5k\alpha_{x\ell_4}^{*}s_{4x}^2+0.5k\alpha_{y\ell_1}s_{1y}^2\right)\right]\exp\left(D_{x\ell_1}s_{1x}+D_{x\ell_2}s_{2x}+D_{x\ell_3}s_{3x}+D_{x\ell_4}s_{4x}\right)$$
$$\times\exp\left[\frac{jk}{2L}\left(s_{1x}^2-s_{2x}^2+s_{3x}^2-s_{4x}^2-2r_xs_{1x}+2r_xs_{2x}-2r_xs_{3x}+2r_xs_{4x}\right)\right]$$
$$\times\exp\left[-\frac{j}{\rho_{\chi s}^2}\left(s_{1x}^2-s_{2x}^2+s_{3x}^2-s_{4x}^2-2s_{1x}s_{3x}+2s_{2x}s_{4x}\right)\right]\left\{\exp\left[-\frac{1}{\rho_t^2}\left(s_{1x}^2+s_{2x}^2+s_{3x}^2+s_{4x}^2-2s_{1x}s_{2x}+2s_{1x}s_{3x}-2s_{1x}s_{4x}-2s_{2x}s_{3x}+2s_{2x}s_{4x}-2s_{3x}s_{4x}\right)\right]\right.$$
$$+2\sigma_\chi^2\exp\left[-\frac{1}{\rho_t^2}\left(2s_{1x}^2+s_{2x}^2+2s_{3x}^2+s_{4x}^2-2s_{1x}s_{2x}-2s_{1x}s_{4x}-2s_{2x}s_{3x}+2s_{2x}s_{4x}-2s_{3x}s_{4x}\right)\right]$$
$$\left.+2\sigma_\chi^2\exp\left[-\frac{1}{\rho_t^2}\left(s_{1x}^2+2s_{2x}^2+s_{3x}^2+2s_{4x}^2-2s_{1x}s_{2x}+2s_{1x}s_{3x}-2s_{1x}s_{4x}-2s_{2x}s_{3x}-2s_{3x}s_{4x}\right)\right]\right\} \quad (4.16)$$

Now by making an association between the alike integration variables of (HF3) and (HF16), it is possible to construct the following matrices. Keep in mind that for $\mathbf{Q}_x$ matrix we have to prepare slightly different matrices of $\mathbf{Q}_{Ax}$, $\mathbf{Q}_{Bx}$, $\mathbf{Q}_{Cx}$ as apparent from (HF15)

$$
\mathbf{Q}_x = \begin{pmatrix}
0.5k\alpha_{x\ell_1} + \frac{1}{2\sigma_s^2} - \frac{jk}{2L} + \frac{j}{\rho_{\chi s}^2} + \frac{q_{11\rho_t^2}}{\rho_t^2} & -\frac{1}{2\sigma_s^2} - \frac{1}{\rho_t^2} & -\frac{j}{\rho_{\chi s}^2} + \frac{q_{13\rho_t^2}}{\rho_t^2} & -\frac{1}{\rho_t^2} \\
-\frac{1}{2\sigma_s^2} - \frac{1}{\rho_t^2} & 0.5k\alpha_{x\ell_2}^* + \frac{1}{2\sigma_s^2} + \frac{jk}{2L} - \frac{j}{\rho_{\chi s}^2} + \frac{q_{22\rho_t^2}}{\rho_t^2} & -\frac{1}{\rho_t^2} & \frac{j}{\rho_{\chi s}^2} + \frac{q_{24\rho_t^2}}{\rho_t^2} \\
-\frac{j}{\rho_{\chi s}^2} + \frac{q_{13\rho_t^2}}{\rho_t^2} & -\frac{1}{\rho_t^2} & 0.5k\alpha_{x\ell_3} + \frac{1}{2\sigma_s^2} - \frac{0.5jk}{L} + \frac{j}{\rho_{\chi s}^2} + \frac{q_{11\rho_t^2}}{\rho_t^2} & -\frac{1}{2\sigma_s^2} - \frac{1}{\rho_t^2} \\
-\frac{1}{\rho_t^2} & \frac{j}{\rho_{\chi s}^2} + \frac{q_{24\rho_0^2}}{\rho_t^2} & -\frac{1}{2\sigma_s^2} - \frac{1}{\rho_t^2} & 0.5k\alpha_{x\ell_4}^* + \frac{1}{2\sigma_s^2} + \frac{jk}{2L} - \frac{j}{\rho_{\chi s}^2} + \frac{q_{22\rho_t^2}}{\rho_t^2}
\end{pmatrix},
$$

$$
q_{11\rho_t^2} = \begin{cases} 1 & \text{for } \mathbf{Q}_{Ax} \text{ and } \mathbf{Q}_{Cx} \\ 2 & \text{for } \mathbf{Q}_{Bx} \end{cases}, \quad q_{13\rho_t^2} = \begin{cases} 1 & \text{for } \mathbf{Q}_{Ax} \text{ and } \mathbf{Q}_{Cx} \\ 0 & \text{for } \mathbf{Q}_{Bx} \end{cases}, \quad q_{22\rho_t^2} = \begin{cases} 1 & \text{for } \mathbf{Q}_{Ax} \text{ and } \mathbf{Q}_{Bx} \\ 2 & \text{for } \mathbf{Q}_{Cx} \end{cases}, \quad q_{24\rho_t^2} = \begin{cases} 1 & \text{for } \mathbf{Q}_{Ax} \text{ and } \mathbf{Q}_{Bx} \\ 0 & \text{for } \mathbf{Q}_{Cx} \end{cases} \tag{4.17a}
$$

$$
\mathbf{R}_x = \begin{pmatrix} -0.5D_{x\ell_1} + \frac{jkr_x}{2L} & -0.5D_{x\ell_2}^* - \frac{jkr_x}{2L} & -0.5D_{x\ell_3} + \frac{jkr_x}{2L} & -0.5jD_{x\ell_4} - \frac{jkr_x}{2L} \end{pmatrix}
$$

$$
\mathbf{R}_y = \begin{pmatrix} -0.5D_{y\ell_1} + \frac{jkr_y}{2L} & -0.5D_{y\ell_2}^* - \frac{jkr_y}{2L} & -0.5D_{y\ell_3} + \frac{jkr_y}{2L} & -0.5jD_{y\ell_4} - \frac{jkr_y}{2L} \end{pmatrix} \tag{4.17b}
$$

$$
\mathbf{N}_x = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{N}_y = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.17c}
$$

$$
\mathbf{M}_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.17d}
$$

For smaller matrices, $\mathbf{R}, \mathbf{N}, \mathbf{M}$, both x and y parts are shown in (4.17).

Exercise 4.2 : Exercise Now we turn to a MATLAB exercise. On the course webpage you will find the m code Sino_Hyp_Her4HFScin.m. Use this m file to generate scintillation plots for the same Gaussian, cos Gaussian, cosh Gaussian, sine Gaussian, sinh Gaussian and annular Gaussian beams which are displayed in Figs. 1 to 10 of the article entitled, "Scintillation calculations for partially coherent general beams via extended Huygens Fresnel integral and self-designed Matlab function", which is also available on course webpage (Fig. 1 of this article is reproduced in Fig. 4.1 below). Also plot the scintillation index curves for partially coherent versions of these beams. Furthermore, for off-axis positions, where r_x, $r_y \neq 0$, repeat the above.

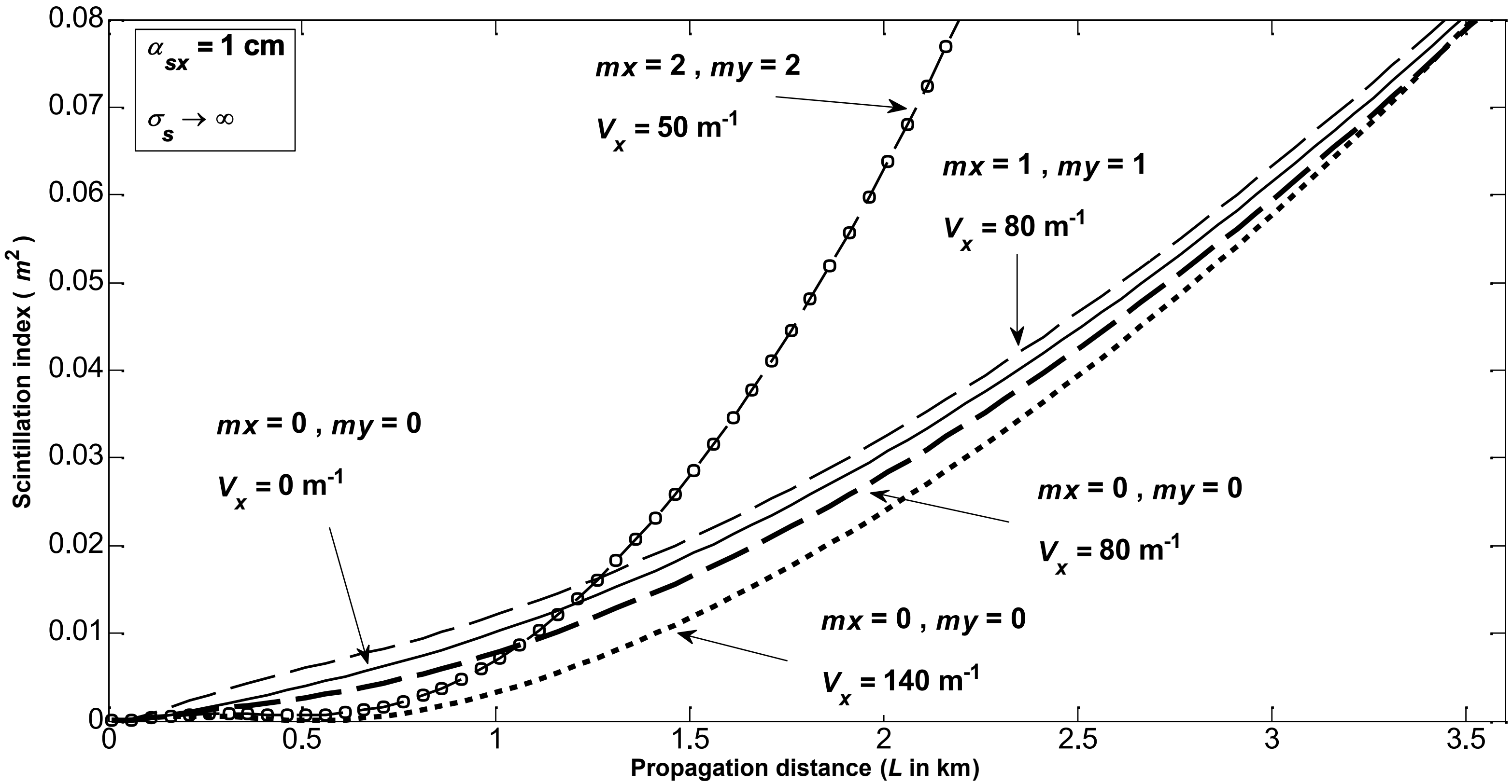

Fig. 4.1 Fig. 1of the article entitled, “Scintillation calculations for partially coherent general beams via extended Huygens Fresnel integral and self-designed Matlab function”.

5. Complex Degree of Coherence

Now we study the complex degree of coherence, $\mu(\ )$ which is expressed by

$$\mu(\mathbf{r}_1,\mathbf{r}_2,L)=\frac{|\Gamma_r(\mathbf{r}_1,\mathbf{r}_2,L)|}{\sqrt{\Gamma_r(\mathbf{r}_1,\mathbf{r}_1,L)}\sqrt{\Gamma_r(\mathbf{r}_2,\mathbf{r}_2,L)}} \tag{5.1}$$

where $\Gamma_r(\ )$ is a two point mutual coherence function. If a source beam having a two point mutual coherence function of $\Gamma_s(\ )$ propagates in turbulence for a distance of L, then $\Gamma_r(\ )$ can be defined as

$$\Gamma_r(\mathbf{r}_1,\mathbf{r}_2,L)=k^2/(2\pi L)^2\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mathrm{d}^2\mathbf{s}_1\mathrm{d}^2\mathbf{s}_2\Gamma_s(\mathbf{s}_1,\mathbf{s}_2)\exp\left\{jk\left[|\mathbf{r}_1-\mathbf{s}_1|^2-|\mathbf{r}_2-\mathbf{s}_2|^2\right]/(2L)\right\}$$
$$\times\exp\left\{-\left[|\mathbf{s}_1-\mathbf{s}_2|^2+(\mathbf{s}_1-\mathbf{s}_2)\bullet(\mathbf{r}_1-\mathbf{r}_2)+|\mathbf{r}_1-\mathbf{r}_2|^2\right]/\rho_t^2\right\} \tag{5.2}$$

Note that in (5.2), in the turbulence exponential, there is the extra term of $(\mathbf{s}_1-\mathbf{s}_2)\bullet(\mathbf{r}_1-\mathbf{r}_2)$, where $\bullet$ signifies dot product, arising since on the left hand side of the equation, mutual coherence function rather than intensity is required. Remember that in intensity expression $\mathbf{r}_1=\mathbf{r}_2$, thus $(\mathbf{s}_1-\mathbf{s}_2)\bullet(\mathbf{r}_1-\mathbf{r}_2)$ equates to zero. In the present study, we take the source to be a partially coherent sinusoidal hyperbolic Gaussian beam in Cartesian coordinates, then from (5.9) of Notes for ECE 635_Eylul 2011, we have

$$\Gamma_s\left(\mathbf{s}_1,\mathbf{s}_2\right)=\Gamma_s\left(s_{1x},s_{1y},s_{2x},s_{2y}\right)=\exp\left(-\frac{s_{1x}^2+s_{2x}^2+s_{1y}^2+s_{2y}^2-2s_{1x}s_{2x}-2s_{1y}s_{2y}}{2\sigma_s^2}\right)\sum_{\ell_1=1}^{N}A_{\ell_1}\exp\left[-\left(0.5k\alpha_{x\ell_1}s_{1x}^2-D_{x\ell_1}s_{1x}\right)\right]\exp\left[-\left(0.5k\alpha_{y\ell_1}s_{1y}^2-D_{y\ell_1}s_{1y}\right)\right]$$

$$\times\sum_{\ell_2=1}^{N}A_{\ell_2}^{*}\exp\left[-\left(0.5k\alpha_{x\ell_2}^{*}s_{2x}^2-D_{x\ell_2}^{*}s_{2x}\right)\right]\exp\left[-\left(0.5k\alpha_{y\ell_2}^{*}s_{2y}^2-D_{y\ell_2}^{*}s_{2y}\right)\right] \quad (5.3)$$

Inserting (5.3) into (5.2), we get

$$\Gamma_r\left(\mathbf{r}_1,\mathbf{r}_2,L\right)=0.5b^2\exp\left[jb\left(r_{1x}^2-r_{2x}^2+r_{1y}^2-r_{2y}^2\right)\right]\exp\left[-\left(r_{1x}-r_{2x}+r_{1y}-r_{2y}\right)^2/\rho_t^2\right]\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}A_{\ell_1}A_{\ell_2}^{*}\frac{\sigma_s^2\rho_t^2}{\left(P_xP_y\right)^{0.5}}E_xE_y \quad (5.4)$$

The various terms in (CDC4) are defined as follows (for x part only)

$$b=k/\left(2L\right),\ E_x=\exp\left\{\left[0.5\sigma_s^2\rho_t^2P_x\left(-r_{1x}+r_{2x}-2jb\rho_t^2r_{1x}+\rho_t^2D_{x\ell_1}\right)^2+Q_x^2\right]/\left(4\rho_t^4C_xP_x\right)\right\}$$

$$P_x=0.5k\left[0.25k\alpha_{x\ell_1}\alpha_{x\ell_2}^{*}\sigma_s^2\rho_t^2+\left(\alpha_{x\ell_1}+\alpha_{x\ell_2}^{*}\right)\left(0.5\sigma_s^2+0.25\rho_t^2\right)+0.5jb\sigma_s^2\rho_t^2\left(\alpha_{x\ell_1}-\alpha_{x\ell_2}^{*}\right)+b^2\sigma_s^2\rho_t^2/k\right]$$

$$Q_x=\left(r_{1x}-r_{2x}\right)\left(-1.5jb\sigma_s^2\rho_t^2-0.5jb\rho_t^4+0.25k\alpha_{x\ell_1}\sigma_s^2\rho_t^2\right)+0.5jb\sigma_s^2\rho_t^4\left(k\alpha_{x\ell_1}-2jb\right)r_{2x}+\rho_t^2D_{x\ell_2}^{*}C_x+\rho_t^2D_{x\ell_1}\left(0.5\sigma_s^2+0.25\rho_t^2\right)$$

$$C_x=0.25k\alpha_{x\ell_1}\sigma_s^2\rho_t^2+0.5\sigma_s^2+0.25\rho_t^2-0.5jb\sigma_s^2\rho_t^2 \quad (5.5)$$

After evaluating $\Gamma_r\left(\mathbf{r}_1,\mathbf{r}_2,L\right)$ as shown in (CDC4), we can retrieve $\Gamma_r\left(\mathbf{r}_1,\mathbf{r}_1,L\right)$ and $\Gamma_r\left(\mathbf{r}_2,\mathbf{r}_2,L\right)$ simply by equating $\mathbf{r}_1$ and $\mathbf{r}_2$. The relevant m code is contained in ComplexD_coh_ECE646.m.

Lab Exercise

1) Verify the derivation (5.4)
2) By running the code given in ComplexD_coh_ECE646.m file, match your figures pointwise with those of Figs. 1, 2, 3, 4, 6, 7, 8, 9 of the article, H. T. Eyyuboğlu, Y. Baykal ve Y. Cai, "Complex degree of coherence for partially coherent general beams in atmospheric turbulence", *Journal of the Optical Society of America A (JOSA A)*, **24**(9), 2891-2901 (2007) whose pdf copy is given on the course webpage.
3) Note that in the present setting, ComplexD_coh_ECE646.m runs for five beams, namely cosh Gaussian beam, cos Gaussian beam, Gaussian beam, annular Gaussian beam and higher order Gaussian beam. The last beam is not covered in the present formulation. Replace this last beam settings alternately by sine and sinh Gaussian beams, taking into account the zero crossings of these beams at on-axis point.
4) Prepare your report and hand it to me in paper format.