Scintillation Formulation via Rytov HTE – 9.04.2009 (revised on 20.06.2011)

Source beam formulations, notations, conversion rules between cylindrical and Cartesian coordinates are included at the end of the document.

The formulation below is based on Bessel Gaussian beam of zeroth order.

Step 1 : We take the source beam, i.e. at the axial point of $z = 0$ expressed either in cylindrical or Cartesian coordinates

Example : Let source beam be a Bessel Gaussian beam of zeroth order, written in cylindrical coordinates as

$$u_s\left(s,\phi_s,z=0\right)=A_c J_0\left(a_B s\right)\exp\left(-k\alpha s^2\right) \qquad (1)$$

where s and ϕ_s are the source plane radial and angular coordinates, A_c defines the amplitude factor, $J_0(\ )$ is the first kind Bessel function having zeroth order, z is the distance measured on the propagation axis, a_B is the width parameter, $k=2\pi/\lambda$ is the wave number with λ being the wavelength, $\alpha=1/(k\alpha_s^2)+0.5j/F_s$ where α_s and F_s respectively refer to radial Gaussian source size and focusing parameter, $j=\sqrt{-1}$.

Notes : 1) In this case A_c is unimportant since, it will cancel out in the further equation development. But the individual amplitude factors will be important for beams containing more than one term, such as annular, dark hollow etc, thus must be retained.

2) In the definition of $\alpha=1/(k\alpha_s^2)+0.5j/F_s$, the scaling of Andrews is adopted. This is compatible with most of our existing publications.

3) To have an idea for the profile, we can simply say that the larger values of width parameter a_B create more Bessel like appearance. For exact profile shapes, refer to our existing publications.

4) In the case of Bessel Gaussian beam of zeroth order, there is no angular variation, so the variable ϕ_s is dummy. It is retained for generality.

Step 2 : Using Huygens-Fresnel integral (not the extended version), the free space receiver field is found. In this particular case of cylindrical coordinates, this is in the form of

$$u_r\left(r,\phi_r,z=L\right)=\frac{-jk\exp\left(jkL\right)}{2\pi L}\int_0^{\infty}\int_0^{2\pi}dsd\phi_s s u_s\left(s,\phi_s,z=0\right)\exp\left\{\frac{jk}{2L}\left[-2rs\cos\left(\phi_r-\phi_s\right)+s^2+r^2\right]\right\} \quad (2)$$

where r and ϕ_r are the receiver plane radial and angular coordinates, L is the axial distance from source plane to receiver plane.

When the sample case of $u_s\left(s,\phi_s,z=0\right)$ is inserted from (1) into (2), $u_r\left(r,\phi_r,z=L\right)$ will become

$$u_r\left(r,\phi_r,z=L\right)=\frac{A_c\exp\left(jkL\right)}{1+2j\alpha L}\exp\left[-\frac{ja_B^2L+2\alpha k^2r^2}{2k\left(1+2j\alpha L\right)}\right]J_0\left(\frac{a_Br}{1+2j\alpha L}\right) \quad (3)$$

Notes : 1) Again ϕ_r is expected to be dummy.

2) The integration in (2) is solved with the help of combination of Eqs. 3.937.1 and 2 and Eq. 6.633.2 or 4 of Gradshteyn and Ryzhik Table of Integrals book (soft copy available from me), these are

$$\int_0^{2\pi}dx\exp\left(p\cos x+q\sin x\right)\exp\left(-jmx\right)=2\pi\frac{\left(p-jq\right)^m}{\left(p^2+q^2\right)^{m/2}}I_m\left(\sqrt{p^2+q^2}\right) \quad (4)$$

$$\int_0^{2\pi}d\phi_s\exp\left[-ja\cos\left(\phi_s-\phi_r\right)\right]\exp\left(-jmx\right)=2\pi\left(-1\right)^m I_m\left(ja\right)\exp\left(-jm\phi_r\right)=2\pi\left(-j\right)^m J_m\left(a\right)\exp\left(-jm\phi_r\right) \quad (5)$$

$$\int_0^{\infty}dx\exp\left(-ax^2\right)J_m\left(px\right)I_m\left(qx\right)x=\frac{1}{2a}\exp\left(\frac{q^2-p^2}{4a}\right)J_m\left(\frac{pq}{2a}\right) \quad (6)$$

Step 3 : Using $u_r(r,\phi_r,z=L)$ as defined in (3) and $u_r(r_1,\phi_1,z=\eta)$ [with r,ϕ_r and L simply replaced by r_1,ϕ_1 and η in (3)], now compute a function $H(r,\phi_r,\kappa,\phi_\kappa,\eta)$ from

$$H(r,\phi_r,\kappa,\phi_\kappa,\eta)=\frac{k^2\exp\left[jk(L-\eta)\right]}{2\pi(L-\eta)u_r(r,\phi_r,z=L)}\exp\left[\frac{jkr^2}{2(L-\eta)}\right]\int_0^\infty\int_0^{2\pi}dr_1d\phi_1r_1u_r(r_1,\phi_1,z=\eta)\exp\left[j\kappa r_1\cos(\phi_1-\phi_\kappa)\right]$$
$$\times\exp\left\{\frac{jk}{2(L-\eta)}\left[r_1^2-2r_1r\cos(\phi_1-\phi_r)\right]\right\} \tag{7}$$

where κ and ϕ_κ indicate the magnitude and the angular orientation of spatial frequency.

Inserting different forms of (3) into (7), for the sample case of Bessel Gaussian beam of zeroth order, $H(r,\phi_r,\kappa,\phi_\kappa,\eta)$ will become

$$H(r,\phi_r,\kappa,\phi_\kappa,\eta)=jk\exp\left[\frac{j\kappa r(1+2j\alpha\eta)}{1+2j\alpha L}\cos(\phi_\kappa-\phi_r)\right]\exp\left[-\frac{0.5j\kappa^2(1+2j\alpha\eta)(L-\eta)}{k(1+2j\alpha L)}\right]$$
$$\times J_0\left\{\frac{a_B\left[\kappa^2(L-\eta)^2-2k\kappa r(L-\eta)\cos(\phi_\kappa-\phi_r)+k^2r^2\right]^{1/2}}{k(1+2j\alpha L)}\right\}\Big/J_0\left(\frac{a_Br}{1+2j\alpha\eta}\right) \tag{8}$$

Notes : The integral in (7) is solved again using (4) and (6)

Step 4 : Scintillation index $m^2(r,\phi_r,L)$ of a single position marked with coordinates of r and ϕ_r on a receiver plane which is $z = L$ away from the source plane is given by

$$m^2(r,\phi_r,L) = 4\pi \int_0^L d\eta \int_0^\infty \kappa d\kappa \int_0^{2\pi} d\phi_\kappa \operatorname{Re}\left[H(r,\phi_r,\kappa,\phi_\kappa,\eta) H^*(r,\phi_r,\kappa,\phi_\kappa,\eta) + H(r,\phi_r,\kappa,\phi_\kappa,\eta) H(r,\phi_r,-\kappa,\phi_\kappa,\eta)\right] \Phi_n(\kappa) \qquad (9)$$

where * is complex conjugate and $\Phi_n(\kappa)$ denotes the specific spectrum that will be used to characterize the turbulent medium. After inserting for $H(\ )$ functions from (8) and by setting $\Phi_n(\kappa)$ to $\Phi_n(\kappa) = 0.033 C_n^2 \exp\left[-\kappa^2(\ell_0/5.92)^2\right] / \left[\kappa^2 + (2\pi/L_0)^2\right]^{11/6}$, i.e. von-Karman spectrum, $m^2(r,\phi_r,L)$, for the sample case of Bessel Gaussian beam of zeroth order, will become

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\begin{aligned}
m^2(r,\phi_r,L) = {} & 0.4147 C_n^2 k^2 \int_0^L d\eta \int_0^\infty \kappa d\kappa \int_0^{2\pi} d\phi_\kappa \frac{\exp\left[-\kappa^2(\ell_0/5.92)^2\right]}{\left[\kappa^2+(2\pi/L_0)^2\right]^{11/6}} \\
& \times \mathrm{Re}\left\{ \exp\left[-\frac{4\alpha_r \kappa r (L-\eta)}{1-4\alpha_i L+4|\alpha|^2 L^2}\cos(\phi_\kappa-\phi_r)\right] \exp\left[-\frac{2\alpha_r \kappa^2 (L-\eta)^2}{k\left(1-4\alpha_i L+4|\alpha|^2 L^2\right)}\right] \right. \\
& \times J_0\left\{ \frac{a_B\left[\kappa^2(L-\eta)^2-2k\kappa r(L-\eta)\cos(\phi_\kappa-\phi_r)+k^2r^2\right]^{1/2}}{k(1+2j\alpha L)} \right\} \\
& \times J_0\left\{ \frac{a_B\left[\kappa^2(L-\eta)^2-2k\kappa r(L-\eta)\cos(\phi_\kappa-\phi_r)+k^2r^2\right]^{1/2}}{k(1-2j\alpha^* L)} \right\} \Big/ \left[J_0\left(\frac{a_B r}{1+2j\alpha\eta}\right) J_0\left(\frac{a_B r}{1-2j\alpha^*\eta}\right)\right] \\
& - \exp\left[-\frac{j\kappa^2(1+2j\alpha\eta)(L-\eta)}{k(1+2j\alpha L)}\right] J_0\left\{ \frac{a_B\left[\kappa^2(L-\eta)^2-2k\kappa r(L-\eta)\cos(\phi_\kappa-\phi_r)+k^2r^2\right]^{1/2}}{k(1+2j\alpha L)} \right\} \\
& \left. \times J_0\left\{ \frac{a_B\left[\kappa^2(L-\eta)^2+2k\kappa r(L-\eta)\cos(\phi_\kappa-\phi_r)+k^2r^2\right]^{1/2}}{k(1+2j\alpha L)} \right\} \Big/ \left[J_0\left(\frac{a_B r}{1+2j\alpha\eta}\right)\right]^2 \right\} \qquad (10)
\end{aligned}
$$

where α_r, α_i and $|\alpha|$ respectively correspond to the real and imaginary parts and the absolute value of α parameter.

Notes : As pointed out above, the receiver plane angular coordinate ϕ_r is dummy and can be taken as zero. Despite this simplification, the integral in (10) does not reduce to double or single integral, so numeric triple integration has to be employed to get results (we have such efficient routines).

Notations and correspondence between cylindrical and Cartesian coordinates

When working in cylindrical coordinates, the fundamental Gaussian beam field expression on the source plane is written as

$$u_s\left(s,\phi_s,z=0\right)=A_c\exp\left(-k\alpha s^2\right) \qquad \text{(N1)}$$

For terms of (N1), the definitions underneath (1) apply. If we switch to a source plane having Cartesian coordinates of s_x and s_y, the field expression of the fundamental Gaussian beam becomes

$$u_s\left(s_x,s_y,z=0\right)=A_c\exp\left[-0.5k\left(\alpha_x s_x^2+\alpha_y s_y^2\right)\right] \qquad \text{(N2)}$$

where $\alpha_x=1/(k\alpha_{sx}^2)+j/F_{sx}$ and $\alpha_y=1/(k\alpha_{sy}^2)+j/F_{sy}$. The correspondence between the two coordinate systems is established via $s^2=s_x^2+s_y^2$, $\alpha_s^2=\alpha_{sx}^2+\alpha_{sy}^2$. If there is $x\,y$ symmetry, that is $\alpha_{sx}=\alpha_{sy}$, then $\alpha_s=\sqrt{2}\alpha_{sx}=\sqrt{2}\alpha_{sy}$ and $\alpha_x=\alpha_y=2\alpha$.

Huygens-Fresnel Receiver and $H\left(r_x,r_y,\kappa,\phi_\kappa,\eta\right)$ Integrals in Cartesian coordinates

$$u_r\left(r_x,r_y,z=L\right)=\frac{-jk\exp\left(jkL\right)}{2\pi L}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}ds_x ds_y u_s\left(s_x,s_y,z=0\right)\exp\left\{\frac{jk}{2L}\left[-2s_x r_x-2s_y r_y+s_x^2+s_y^2+r_x^2+r_y^2\right]\right\} \qquad \text{(N3)}$$

$$H\left(r_x,r_y,\kappa,\phi_\kappa,\eta\right)=\frac{k^2\exp\left[jk\left(L-\eta\right)\right]}{2\pi\left(L-\eta\right)u_r\left(r_x,r_y,z=L\right)}\exp\left[\frac{jk\left(r_x^2+r_y^2\right)}{2(L-\eta)}\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dr_{1x}dr_{1y}u_r\left(r_{1x},r_{1y},z=\eta\right)\exp\left\{j\kappa\left[r_{1x}\cos\left(\phi_\kappa\right)+r_{1y}\sin\left(\phi_\kappa\right)\right]\right\}$$
$$\times\exp\left[\frac{jk}{2(L-\eta)}\left(r_{1x}^2+r_{1y}^2-2r_x r_{1x}-2r_y r_{1y}\right)\right] \qquad (N4)$$

Source beam formulations

1) Laguerre-Gaussian

A) In cylindrical coordinates

$$u_s\left(s,\phi_s,z=0\right)=A_c\left(\frac{\sqrt{2}s}{\alpha_s}\right)^m(-j)^m\exp\left(jm\phi_s\right)\exp\left(-k\alpha s^2\right)L_n^m\left(\frac{2s^2}{\alpha_s^2}\right) \quad \text{(S1)}$$

B) In Cartesian coordinates

$$u_s\left(s_x,s_y,z=0\right)=A_c\left[\frac{\left(s_x+js_y\right)}{\alpha_{sc}}\right]^m(-j)^m\exp\left[-0.5k\left(\alpha_x s_x^2+\alpha_y s_y^2\right)\right]L_n^m\left(\frac{s_x^2}{\alpha_{sx}^2}+\frac{s_y^2}{\alpha_{sy}^2}\right) \quad \text{(S2)}$$

where $\alpha_{sc}=\alpha_{sx}=\alpha_{sy}$

2) Laser array

A) In Cartesian coordinates

$$u_s\left(s_x,s_y,z=0\right)=\sum_{n=1}^{N}\exp\left[-\frac{0.5}{\alpha_{sc}^2}\left(s_x^2+s_y^2-2r_0s_x\cos\varphi_n-2r_0s_y\sin\varphi_n+r_0^2\right)\right] \quad \text{(S3)}$$

where r_0 is the radial distance of beamlet, φ_n is the angular position of nth beamlet, for illustration see FIGURE 1 in

H. T. Eyyuboğlu, Y. Baykal and Y. Cai, "Scintillations of laser array beams", *Journal of Optics A: Applied Physics B - Laser and Optics* **91**(2), 265-271 (2008).

3) Bessel-Gaussian

A) In cylindrical coordinates

$$u_s\left(s,\phi_s,z=0\right)=A_c J_n\left(a_B s\right)\exp\left(-k\alpha s^2\right)\exp\left(-jn\phi_s\right) \quad \text{(S4)}$$

4) Modified Bessel-Gaussian

A) In cylindrical coordinates

$$u_s\left(s,\phi_s,z=0\right)=A_c I_n\left(a_B s\right)\exp\left(-k\alpha s^2\right)\exp\left(-jn\phi_s\right) \quad \text{(S5)}$$

5) Higher order dark hollow

A) In Cartesian coordinates

$$u_s\left(s_x,s_y,z=0\right)=H_n\left(a_x s_x+b_x\right)H_m\left(a_y s_y+b_y\right)\sum_{r=1}^{R}\sum_{t=1}^{T}\binom{R}{r}\binom{T}{t}\frac{(-1)^{r+t}}{RT}\left\{A_1\exp\left[-0.5k\left(r\alpha_{x1}s_x^2+t_s\alpha_{y1}s_y^2\right)\right]-A_2\exp\left[-0.5k\left(r\alpha_{x2}s_x^2+t_s\alpha_{y2}s_y^2\right)\right]\right\} \quad \text{(S6)}$$

where a_x and a_y are the width parameters and b_x and b_y are the displacement parameters for the Hermite polynomials $H_n\left(\ \right)$ and $H_m\left(\ \right)$. $\binom{R}{r}$ and $\binom{T}{t}$ are the binomial coefficients associated with the sums taken over *R* and *T* via the indices *r* and *t*. $t_s=t$ when $T>1$ and $t_s=r$ if $T=1$. $\alpha_{x1}=1/\left(k\alpha_{sx1}^2\right)+j/F_{x1}$ with α_{sx1} and F_{x1} are respectively the Gaussian source size and the focusing parameter of the first beam.

Similar definitions are applicable for $\alpha_{x2},\ \alpha_{y1},\ \alpha_{y2}$ provided that the conditions $\alpha_{sx1} > \alpha_{sx2},\ \alpha_{sy1} > \alpha_{sy2}$ are satisfied.

6) Vortex

A) In cylindrical coordinates

$$u_s\left(s,\phi_s,z=0\right)=A_c\exp\left(-k\alpha s^2\right)\exp\left(-jn\phi_s\right) \qquad \text{(S7)}$$

7) Higher order sinusoidal / hyperbolic

A) In Cartesian coordinates

$$u_s\left(s_x,s_y,z=0\right)=\sum_{\ell=1}^{N}A_\ell H_{n_\ell}\left(a_{x\ell}s_x+b_{x\ell}\right)\exp\left[-\left(0.5k\alpha_{x\ell}s_x^2+jV_{x\ell}s_x\right)\right]H_{m_\ell}\left(a_{y\ell}s_y+b_{y\ell}\right)\exp\left[-\left(0.5k\alpha_{y\ell}s_y^2+jV_{y\ell}s_y\right)\right] \qquad \text{(S8)}$$

All ℓ subscripted terms establish the specific parameters of the individual beams comprising the general beam through summation. In this manner, N denotes the number of beams, A_ℓ is the amplitude factor, $H_{n_\ell}\left(a_{x\ell}s_x+b_{x\ell}\right)$ and $H_{m_\ell}\left(a_{y\ell}s_y+b_{x\ell}\right)$ are Hermite polynomials governing the beam variations for s_x and s_y directions, where n_ℓ and m_ℓ are the order, $a_{x\ell}$ and $a_{y\ell}$ characterize the width, $b_{x\ell}$ and $b_{y\ell}$ are the complex displacement parameters, $V_{x\ell}$ and $V_{y\ell}$ are the complex parameters used to create physical location displacement and phase rotation or a combination of both. The way of obtaining sinusoidal / hyperbolic beams from (S8) is explained in Table 1, given in

Ç. Arpali, C. Yazıcıoğlu, H.T. Eyyuboğlu, S. A. Arpali and Y. Baykal, "Simulator for general-type beam propagation in turbulent atmosphere", *Optics Express*, **14**(20), 8918-8928 (2006).

The method of inserting partial coherence property into the above source beam formulations

For any of the source beams above, we define a (spatial) partial coherence parameter σ_s and make the beam partially coherent by inserting it in the mutual coherence function $\Gamma_s(\ )$, which is related to source beam field expressions as follows

A) In cylindrical coordinates

$$\Gamma_s(s_1,\phi_{s1},s_2,\phi_{s2},z=0)=u_s(s_1,\phi_{s1},z=0)u_s^*(s_2,\phi_{s2},z=0)\exp\left\{-0.5\left[-2s_1s_2\cos(\phi_1-\phi_2)+s_1^2+s_2^2\right]/\sigma_s^2\right\} \quad \text{(S9)}$$

where s_1,ϕ_{s1} and s_2,ϕ_{s2} refer to two distinct locations on the source plane.

B) In Cartesian coordinates

$$\Gamma_s(s_{1x},s_{1y},s_{2x},s_{2y},z=0)=u_s(s_{1x},s_{1y},z=0)u_s^*(s_{2x},s_{2y},z=0)\exp\left\{-0.5\left[-2s_{1x}s_{2x}-2s_{1y}s_{2y}+s_{1x}^2+s_{1y}^2+s_{2x}^2+s_{2y}^2\right]/\sigma_s^2\right\} \quad \text{(S10)}$$

where s_{1x},s_{1y} and s_{2x},s_{2y} refer to two distinct locations on the source plane.